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A TREATISE

ON

HYDRODYNAMICS

With numerous Examples.

BY

A. B. BASSET, M.A.

OF LINCOLN'S INN, BARRISTER AT LAW; FELLOW OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY; AND FORMERLY SCHOLAR OF TRINITY COLLEGE, CAMBRIDGE.

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PREFACE.

In the present Treatise I have endeavoured to lay before the reader in a connected form, the results of the most important investigations in the mathematical theory of Hydrodynamics, which have been made during modern times. The Science of Hydrodynamics may properly be considered to include an enquiry into the motion of all fluids, gaseous as well as liquid; but for reasons which are stated in the introductory paragraph of Chapter I., the present treatise is confined almost entirely to the motion of liquids. The progress of scientific knowledge in all its branches has been the peculiar feature of the present century, and it is therefore not surprising that during the last fifty years a great increase in hydrodynamical knowledge has taken place; but many of the most important results of writers upon this subject have never been inserted in any treatise, and still lie buried in a variety of British and foreign mathematical periodicals and transactions of learned Societies; and it has been my aim to endeavour to collect together those investigations which are of most interest to the mathematician, and to condense them into a form suitable for a treatise.

The present work is divided into two volumes, the first of which deals with the theory of the motion of frictionless liquids, up to and including the theory of the motion of solid bodies in a liquid. In the second volume, a considerable portion of which is already written, it is proposed to discuss the theory of rectilinear and circular vortices; the motion of a liquid ellipsoid

under the influence of its own attraction, including Professor G. H. Darwin's important memoir on dumb-bell figures of equilibrium; the theories of liquid waves and tides; and the theory of the motion of a viscous liquid and of solid bodies therein.

References have been given throughout to the original authorities which have been incorporated or consulted; and a collection of examples has been added, most of which have been taken from University or College Examination Papers, which have been set during recent years.

The valuable report of Mr W. M. Hicks on Hydrodynamics, to the British Association in 1881—2, has proved of great service in the difficult task of collecting and arranging materials. I have also to express my obligations to the English treatises of Dr Besant and Professor Lamb, from the latter of which I have received considerable assistance in Chapters IV. and VI.; and also to the German treatise of the late Professor Kirchhoff.

I am greatly indebted to Professor Greenhill for his kindness in having read the proof sheets, and also for having made many valuable suggestions during the progress of the work.

In a treatise which contains a large amount of analytical detail, it is probable that there are several undetected errors; and I shall esteem it a favour if those of my readers who discover any errors or obscurities of treatment, or have any suggestions to make, will communicate with me.

United University Club,
Pall Mall, East.

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ERRATA.

Page 11 line 7 *read*, $\iint(lu+mv+nw)dS$.
" 15 " 11 " where for when.
" 31 § 35 " fluid for liquid.
" 32 line 11 *add*, and taking account of (21).
" 39 " 20 *read*, v for V.
" 46 " 8 " Π for π.
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" 54 In the figure *read*, $OR'H'S'$ for $OR'H'S$.
" 55 line 21 *read*, strength for density.
" 68 § 63 *read*, fluid for liquid.
" 70 § 68 Insert the letter D in the figure.
" 80 line 14 *read*, $\pi\rho$ for $2\pi\rho$.
" 107 " 2 from bottom *read*, $\frac{1}{2}\pi L_{n+1}$ for $\frac{1}{2}L_{n+1}$.
" 108 " 8, 14, 18, 19, 20 *read*, $2E$ and $2F$ for E and F.
" 110 " 8 *read*, $\frac{1}{4}\pi$ for $\frac{1}{2}\pi$.
" 114 multiply the values of P, Q and L by ρ.
" 116 line 16 *add*, with unit velocity.
" 163 " 11 and p. 170 line 15 *read*, impulsive pressure.
" 187 " 19 *read*, terms in p/ρ.
" 195 " 5 " is for be.
" 202 " 1 and 2 from bottom *read*, A for a.
" 221 " 13 *read*, ratio of the terminal to the initial velocity.
" 223 In the figure *read*, y for η.
" 241 footnote *read*, Kugelfunctionen.
" 247 line 13 *read*, $2B/C$, B^2/C^2 for $2B/c$, B^2/c^2.

CHAPTER I.

HYDROKINEMATICS.

1. THE science of Hydrodynamics may be divided into two separate branches, viz. the motion of liquids and the motion of gases. The chief interest arising from the latter branch of the subject is due to the fact that air is the vehicle by means of which sound is transmitted, and consequently the discussion of special problems relating to the motion of gases belongs to the theory of sound rather than to hydrodynamics; it must also be recollected that in order to deal satisfactorily with many problems connected with the motion of gases, it is necessary to take into account changes of temperature and other matters which properly belong to the science of thermodynamics. In the earlier chapters of the present treatise the general theory of the motion of fluids is discussed, including those peculiarities of motion which are alike common to liquids and gases; but the subsequent chapters are limited almost entirely to the consideration of special problems relating to the motion of liquids.

In ancient times very little advance in hydrodynamics appears to have been made. In modern times the earliest pioneers were Torricelli and Bernoulli, whose investigations were due to the hydraulic requirements of Italian ornamental landscape gardening; but the first great step was taken by D'Alembert and Euler, who in the last century successfully applied dynamical principles to the subject, and thereby discovered the general equations of motion of a perfect fluid, and placed the subject on a satisfactory basis. The discovery of the general equations of motion was followed up by the investigations of the great French mathematicians Laplace, Lagrange and Poisson, the first of whom has left us a splendid memorial of his genius in his celebrated *Theory of the Tides*.

The next advance was made by Poisson[1] and Green[2]; the former of whom in 1831 discovered the velocity potential due to the motion of a sphere in an unlimited liquid, and the latter of whom in 1833, without a knowledge of Poisson's work, discovered the velocity potential due to the motion of translation of an ellipsoid in an unlimited liquid. Green's investigation was completed for the case of rotation by Clebsch[3] in 1856.

The velocity potential due to the motion of a variety of cylindrical surfaces has also been discovered during the last fifteen years; but a similar advance has not been made as regards the motion of two or more solids. The kinetic energy of a liquid due to the motion of two cylinders whose cross sections are circular, has been obtained by Hicks[4] and Greenhill[5]. The former has also written several valuable papers on the motion of two spheres[6], which have placed this problem in a perfectly satisfactory condition. A complete discussion of the motion of two oblate or prolate spheroids whose excentricities are nearly equal to zero or unity, would be an attractive subject for investigation, and would throw light on the motion of two ships sailing alongside one another.

In 1845 Professor Stokes[7] published his well-known theory of the motion of a viscous liquid, in which he endeavoured to account for the frictional action which exists in all known liquids, and which causes the motion to gradually subside by converting the kinetic energy into heat. This paper was followed up in 1850 by another[8], in which he solved various problems relating to the motion of spheres and cylinders in a viscous liquid. Previously to this paper no problem relating to the motion of a solid body in a liquid had ever been solved, in which the viscosity had been taken into account.

Since the time of Lagrange the essential difference between the motion of a fluid when a velocity potential exists and when it does not exist had been recognised; and an opinion very generally

[1] *Mém. de l'Acad. des Sciences.* Paris, vol. XI. p. 521.

[2] *Trans. Roy. Soc. Edinburgh*, vol. XIII. p. 54.

[3] *Crelle*, vol. LII. p. 119.

[4] *Quart. Journ.*, vol. XVI. pp. 113 and 193.

[5] *Ibid.* vol. XVIII. pp. 356—362.

[6] *Proc. Camb. Phil. Soc.*, vol. III. p. 276, vol. IV. p. 29, and *Phil. Trans.*, 1880.

[7] *Trans. Camb. Phil. Soc.*, vol. VIII. p. 287.

[8] *Ibid.* vol. IX. part II. p. 8.

prevailed that if at any particular instant some particular portion of the fluid were moving in such a manner that a velocity potential existed, the subsequent motion of this same portion of fluid would always be such that the component velocities of its elements would be derivable from a velocity potential. The first rigorous proof of this important proposition was given by Cauchy, and a different one was subsequently given by Stokes[1], but until the year 1858 no complete investigation respecting the peculiarities of rotational motion had ever been made. This was effected by Helmholtz[2] in his celebrated memoir on Vortex Motion, which may perhaps be considered the most important step in hydrodynamics which has been made during the present century. The same subject was subsequently taken up by Sir W. Thomson[3] and the theory of polycyclic velocity potentials fully investigated. During the last six years important additional investigations on the theory of vortex rings have been made by Hicks[4] and J. J. Thomson[5].

The last twenty years have witnessed a great advance in hydrodynamics, and numerous important papers have been written by many eminent mathematicians both British and foreign, which will be considered in detail in the present work.

We shall now proceed to consider the definitions and principles of the subject.

2. A fluid may be defined to be an aggregation of molecules, which yield to the slightest effort made to separate them from each other, if it be continued long enough. All fluids with which we are acquainted may be divided into liquids and gases; the former are so slightly compressible that they are usually regarded as incompressible fluids, whilst the latter are very highly compressible.

A *perfect* fluid is one which is incapable of sustaining any tangential stress or action in the nature of a shear; and it will be shown in the next chapter that the consequence of this property is, that the pressure at every point of a perfect fluid is equal in all directions, whether the fluid be at rest or in motion. A

[1] *Trans. Camb. Phil. Soc.*, vol. VIII. p. 305.

[2] *Crelle*, vol. LV. p. 25; translated by Tait, *Phil. Mag.* (4) XXXIII. p. 485.

[3] *Trans. Roy. Soc. Edin.*, vol. XXV. p. 217.

[4] *Phil. Trans.*, 1881, 1884 and 1885.

[5] *Adams' Prize Essay*, 1882.

perfect fluid is however an entirely ideal substance, since all fluids with which we are acquainted are capable of offering resistance to tangential stresses. This property, which is known as viscosity, gives rise to an action in the nature of friction, by which the kinetic energy is gradually converted into heat.

In the case of gases, water and many other liquids, the effects of viscosity are small; such fluids may therefore be approximately regarded as perfect fluids. It will therefore be desirable to commence with the study of the motion of perfect fluids, reserving the consideration of viscous fluids for the second volume.

There are certain kinematical propositions which are true for all fluids, and which it will be convenient to investigate before entering upon the dynamical portion of the subject. These propositions form the subject of the present chapter.

3. The motion of a fluid may be investigated by two different methods, the first of which is called the Lagrangian method, and the second the Eulerian or flux method, although both are due to Euler.

In the Lagrangian method, we fix our attention upon an element of fluid, and follow its motion throughout its history. The variables in this case are the initial coordinates a, b, c of the particular element upon which we fix our attention, and the time. This method has been successfully employed in the solution of very few problems.

In the Eulerian or flux method, we fix our attention upon a particular point of the space occupied by the fluid, and observe what is going on there. The variables in this case are the coordinates x, y, z of the particular point of space upon which we fix our attention, and the time.

Velocity and Acceleration.

4. In forming expressions for the velocity and acceleration of a fluid, it is necessary to carefully distinguish between the Langrangian and the flux method.

I. *The Langrangian Method.*

Let u, v, w be the component velocities parallel to fixed axes, of an element of fluid whose coordinates are x, y, z and $x + \delta x$, $y + \delta y$, $z + \delta z$ at times t and $t + \delta t$ respectively, then

$$u = dx/dt = \dot{x},\ v = \dot{y},\ w = \dot{z} \quad\text{..................(1)},$$

where in forming $\dot{x}$, $\dot{y}$, $\dot{z}$ we must suppose x, y, z to be expressed in terms of the initial coordinates a, b, c and the time.

If the axes, instead of being fixed, were moving with angular velocities θ_1, θ_2, θ_3 about themselves, the component velocities would be given by the equations,

$$u = \dot{x} - y\theta_3 + z\theta_2,\ v = \dot{y} - z\theta_1 + x\theta_3,\ w = \dot{z} - x\theta_2 + y\theta_1 \ldots (2).$$

It should be noticed that $\dot{x}$, $\dot{y}$, $\dot{z}$ are the velocities of the fluid *relative* to the moving axes.

The expressions for the component accelerations are

$$f_x = \dot{u} = \ddot{x},\ f_y = \ddot{y},\ f_z = \ddot{z} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

when the axes are fixed, and

$$f_x = \dot{u} - v\theta_3 + w\theta_2,\ f_y = \dot{v} - w\theta_1 + u\theta_3,\ f_z = \dot{w} - u\theta_2 + v\theta_1 \ldots (4)$$

when the axes are in motion. Here u, v, w must be supposed to be expressed in terms of a, b, c and t.

II. *The Flux Method.*

5. Let δQ be the quantity of fluid which in time δt flows across any small area A, which passes through a fixed point P in the fluid; let ρ be the density of the fluid, q its resultant velocity, and ϵ the angle which the direction of q makes with the normal to A, drawn towards the direction in which the fluid flows. Then

$$\delta Q = \rho q A \delta t \cos \epsilon,$$

therefore

$$q = \frac{1}{\rho A \cos \epsilon} \frac{dQ}{dt}.$$

Now $A \cos \epsilon$ is the projection of A upon a plane passing through P perpendicular to the direction of motion of the fluid; hence δQ is the independent of the direction of the area, and is the same for all areas whose projections upon the above-mentioned plane are equal. Hence the velocity is equal to the rate per unit of area divided by the density, at which liquid flows across a plane perpendicular to its direction of motion.

The velocity is therefore a function of the position of P and the time.

6. We may therefore put $u = F(x, y, z, t)$; whence if the axes are fixed, and if $u + \delta u$ be the velocity parallel to x at time $t + \delta t$ of the element of fluid which at time t was situated at the point (x, y, z),

$$\delta u = F(x + u\delta t,\ y + v\delta t,\ z + w\delta t,\ t + \delta t) - F(x, y, z, t).$$

Therefore the acceleration,

$$f_x = \lim \frac{\delta u}{\delta t} = \frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}.$$

Hence if $\partial/\partial t$ denotes the operator

$$d/dt + ud/dx + vd/dy + wd/dz,$$

the component accelerations will be given by the equations

$$f_x = \frac{\partial u}{\partial t},\ f_y = \frac{\partial v}{\partial t},\ f_z = \frac{\partial w}{\partial t} \quad \text{..................(5).}$$

When the axes are in motion let $u + \delta u$ be the component velocity at time $t + \delta t$, parallel to the new position of the axis of x, of the element which at time t was situated at the point x, y, z; then if U, V, W be the component velocities *relative* to the axes,

$$\delta u = F(x + U\delta t,\ y + V\delta t,\ z + W\delta t,\ t + \delta t) - F(x, y, z, t).$$

Therefore

$$\frac{\delta u}{\delta t} = \frac{du}{dt} + U\frac{du}{dx} + V\frac{du}{dy} + W\frac{du}{dz},$$

where the values of U, V, W are given by (2). Hence if $\partial/\partial t$ denote the operator $d/dt + Ud/dx + Vd/dy + Wd/dz$, the component accelerations parallel to the moving axes are given by the equations

$$f_x = \frac{\partial u}{\partial t} - v\theta_3 + w\theta_2,\ f_y = \frac{\partial v}{\partial t} - w\theta_1 + u\theta_3,\ f_z = \frac{\partial w}{\partial t} - u\theta_2 + v\theta_1 \ldots\text{(6).}$$

Similarly it can be shown that if ϖ, θ, z be cylindrical coordinates, and u, v, w be the component velocities measured in the directions in which the former quantities increase,

$$f_\varpi = \frac{\partial u}{\partial t} - \frac{v^2}{\varpi},\ f_\theta = \frac{\partial v}{\partial t} + \frac{uv}{\varpi},\ f_z = \frac{\partial w}{\partial t} \quad \text{............(7),}$$

where

$$\frac{\partial}{\partial t} = \frac{d}{dt} + u\frac{d}{d\varpi} + \frac{v}{\varpi}\frac{d}{d\theta} + w\frac{d}{dz}.$$

If (r, θ, ϕ) be polar coordinates and u, v, w be the velocities measured in the directions in which these quantities increase,

$$f_r = \frac{\partial u}{\partial t} - \frac{v^2 + w^2}{r},\ f_\theta = \frac{\partial v}{\partial t} + \frac{uv}{r} - \frac{w^2}{r}\cot\theta,$$

$$f_\phi = \frac{\partial w}{\partial t} + \frac{uw}{r} + \frac{uv}{r}\cot\theta \text{.........(8),}$$

where

$$\frac{\partial}{\partial t} = \frac{d}{dt} + u\frac{d}{dr} + \frac{v}{r}\frac{d}{d\theta} + \frac{w}{r\sin\theta}\frac{d}{d\phi}.$$

The Equation of Continuity.

7. Before proceeding further, it will be convenient to introduce the following lemma, which is a particular case of Green's Theorem, which will be considered more fully in Chapter IV.

Let ξ, η, ζ be any functions of x, y, z, which are finite and continuous at all points within a closed surface S, then

$$\iiint \left(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\right) dxdydz = \iint (l\xi + m\eta + n\zeta)\, dS \ldots (9),$$

where the triple integral extends throughout the volume enclosed by S, and the double integral is taken over the surface of S, and l, m, n and the direction cosines of the normal at any point of S drawn outwards.

Integrating the left-hand side of (9) by parts we obtain

$$\iiint \frac{d\xi}{dx}\, dxdydz = \left[\iint \xi\, dydz\right],$$

where the brackets refer to the limits of integration. Now since the surface S is closed, it follows that any line parallel to x which enters the surface a given number of times must issue from it the same number of times, hence if l is positive at the point of entrance, it must be negative at the corresponding point of exit; hence

$$[\iint \xi\, dydz] = \iint l\, \xi\, dS,$$

where the integration with respect to S extends over the whole surface. Treating the other two terms in a similar manner we obtain the theorem in question.

8. If the motion of a fluid be continuous, it is evident that the increase in the amount of fluid within a fixed space, which takes place during any given interval, must be equal to the amount which flows in across the boundaries of that space.

Let ρ be the density of the fluid at time t, then the increment during an interval δt in the mass of the fluid bounded by any fixed surface S,

$$= \iiint \frac{d\rho}{dt}\; \delta t\, dxdydz.$$

The amount of fluid which flows into S across the boundary,

$$= -\iint \rho\,(lu + mv + nw)\, \delta t\, dS,$$

$$= -\iiint \left\{\frac{d(\rho u)}{dx}+\frac{d(\rho v)}{dy}+\frac{d(\rho w)}{dz}\right\} \delta t\, dxdydz,$$

by (9). Equating these two values of the increment, we obtain

$$\frac{d\rho}{dt}+\frac{d(\rho u)}{dx}+\frac{d(\rho v)}{dy}+\frac{d(\rho w)}{dz}=0 \quad\text{............(10).}$$

This equation is usually called the equation of continuity.

In the case of a liquid ρ is constant, whence

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0 \quad\text{......................(11).}$$

9. The same result is often obtained in a different manner, which we shall illustrate by finding the equation of continuity of a liquid referred to polar coordinates.

Let u, v, w be the velocities in the r, θ, ϕ directions, and let $r^2\sin\theta\,\delta r\,\delta\theta\,\delta\phi$ be a small element of volume. The quantity of liquid which in unit of time flows in across the face $r^2\sin\theta\,\delta\theta\,\delta\phi$

$$=\rho u r^2\sin\theta\,\delta\theta\,\delta\phi.$$

The quantity which flows out across the opposite face

$$=\rho u r^2\sin\theta\,\delta\theta\,\delta\phi+\rho\sin\theta\,\frac{d}{dr}(r^2u)\,\delta r\,\delta\theta\,\delta\phi.$$

Hence the total loss

$$=\rho\sin\theta\,\frac{d(r^2u)}{dr}\,\delta r\,\delta\theta\,\delta\phi.$$

Equating the total loss due to the flow across all the faces of the element to zero, we obtain

$$\sin\theta\,\frac{d(r^2u)}{dr}+r\,\frac{d(v\sin\theta)}{d\theta}+r\,\frac{dw}{d\phi}=0 \quad\text{.........(12).}$$

If cylindrical coordinates are employed, the equation is

$$\frac{d(\varpi u)}{d\varpi}+\frac{dv}{d\theta}+\varpi\,\frac{dw}{dz}=0\text{..................(13).}$$

10. In a large and important number of problems the quantity $udx+vdy+wdz$ is a perfect differential $d\phi$, whence

$$u=d\phi/dx,\quad v=d\phi/dy,\quad w=d\phi/dz\,;$$

hence if ds be a linear element drawn in any direction, and q be the velocity in the same direction $q=d\phi/ds$. The function ϕ is called the *velocity potential.*

Substituting the above values of u, v, w in (11), we obtain

$$\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}+\frac{d^2\phi}{dz^2}=0\text{.....................(14),}$$

or

$$\nabla^2\phi=0.$$

This equation is usually known as Laplace's equation, and the operator ∇^2 as Laplace's operator.

The values of ∇^2 in polar and cylindrical coordinates are respectively,

$$\nabla^2 = \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + \frac{1}{r^2}\frac{d^2}{d\theta^2} + \frac{\cot\theta}{r^2}\frac{d}{d\theta} + \frac{1}{r^2\sin^2\theta}\frac{d^2}{d\phi^2} \text{...(15)},$$

and
$$\nabla^2 = \frac{d^2}{d\varpi^2} + \frac{1}{\varpi}\frac{d}{d\varpi} + \frac{1}{\varpi^2}\frac{d^2}{d\theta^2} + \frac{d^2}{dz^2} \text{ ....(16).}$$

These results may be readily obtained by substituting the values of u, v, w in terms of ϕ in (12) and (13).

11. The preceding forms of the equation of continuity are not convenient when the Lagrangian method is employed. To find an appropriate form, consider a small rectangular parallelopiped whose diagonal is PQ. Let a, b, c, $a+\delta a$, $b+\delta b$, $c+\delta c$ be the coordinates of P and Q respectively. At the end of a time t, the fluid of which the parallelopiped is composed will form a differently situated oblique-angled parallelopiped. The volume of the latter $= J\delta a\,\delta b\,\delta c$,

where J is the Jacobian of x, y, z and is equal to

$$\begin{vmatrix} \frac{dx}{da}, & \frac{dy}{da}, & \frac{dz}{da} \\ \frac{dx}{db}, & \frac{dy}{db}, & \frac{dz}{db} \\ \frac{dx}{dc}, & \frac{dy}{dc}, & \frac{dz}{dc} \end{vmatrix}.$$

Hence if ρ_0 be the initial density, and ρ the density at time t, the required equation is

$$J\rho = \rho_0 \text{ ....(17).}$$

In the case of a liquid $\rho = \rho_0$ and therefore

$$J = 1 \text{..............................(18).}$$

The Bounding Surface.

12. Besides the equations which must be satisfied within the interior of a fluid, it is necessary that certain other conditions should be satisfied at the boundary, which depend upon the special problem under consideration.

If the fluid is bounded by a surface whose equation referred to axes fixed in space is $F(x, y, z, t) = 0$, the normal velocity of the

fluid at the surface must be equal to the normal velocity of the surface, hence the sheet of fluid of which the boundary is composed must always consist of the same elements of fluid. Hence

$$F(x+u\delta t,\ y+v\delta t,\ z+w\delta t,\ t+\delta t)=0,$$

and therefore

$$\frac{dF}{dt}+u\frac{dF}{dx}+v\frac{dF}{dy}+w\frac{dF}{dz}=0 \text{ ....(19).}$$

If the boundary is fixed, the condition becomes

$$lu+mv+nw=0. \text{ ....(20).}$$

If the axes be in motion, the condition is

$$\frac{dF}{dt}+U\frac{dF}{dx}+V\frac{dF}{dy}+W\frac{dF}{dz}=0. \text{ ....(21),}$$

where U, V, W are the velocities of an element of fluid *relative* to the axes.

It should be noticed that (19) or (21) must be satisfied by every surface which is composed of the same elements of fluid.

Lines of Flow and Stream Lines.

13. DEF. A *line of flow* is a line whose direction coincides with the direction of the resultant velocity of the fluid.

The differential equations of a line of flow are

$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}.$$

Hence if $\chi_1(x, y, z, t)=\alpha_1$, $\chi_2(x, y, z, t)=\alpha_2$ be any two independent integrals, the equations $\chi_1=\text{const.}$, $\chi_2=\text{const.}$, are the equations of two families of surfaces whose intersections determine the lines of flow.

DEF. A *stream line, or a line of motion,* is a line whose direction coincides with the direction of the actual paths of the elements of fluid.

The equations of a stream line are determined by the simultaneous differential equations,

$$\dot{x}=u,\ \dot{y}=v,\ \dot{z}=w,$$

where x, y, z must be regarded as unknown functions of t. The integration of these equations will determine x, y, z in terms of the initial coordinates and the time.

14. If through every point of a small closed curve lines of flow be drawn, they will enclose a mass of fluid which may be called a tube of flow.

Let us apply the lemma of § 7 to a portion of liquid bounded by a tube of flow and two planes perpendicular to it. Putting $u=\xi$, $v=\eta$, $w=\zeta$, and taking account of (11), we obtain

$$0=\iiint\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)dxdydz=\iint\left(lu+mv+nw\right)dS.$$

At every point of the curved surface of the tube of flow, $lu+mv+nw=0$; at the two ends this quantity is respectively equal to q_1 and $-q_2$, where q_1 and q_2 are the velocities of the liquid at the ends. Hence the surface integral $=q_1dS_1-q_2dS_2=0$; whence *the product of the velocity of a liquid and the cross section of a tube of flow is constant throughout the length of the latter.*

In the next place, *a line of flow cannot begin or end in any portion of a liquid throughout which the velocity is finite, but must either form a closed curve or have its extremities in the boundaries of the portion of liquid.*

For if a line of flow ended the liquid, it would be possible to draw a closed surface cutting a tube of flow once only. Hence $lu+mv+nw$ would be zero at every point of the closed surface excepting where it cuts the tube of flow, and therefore the surface integral would not be zero.

15. When a velocity potential exists, the equation

$$udx+vdy+wdz=0$$

is the equation of a family of surfaces, at every point of which the velocity potential has a definite constant value, and which may be called surfaces of equi-velocity potential.

If P be any point on the surface, $\phi=$ const., and dn be an element of the normal at P which meets the neighbouring surface $\phi+\delta\phi$ at Q, the velocity at P along PQ will be equal to $d\phi/dn$; hence $d\phi$ must be positive, and therefore a fluid always flows from places of lower to places of higher velocity potential.

The lines of flow evidently cut the surfaces of equi-velocity potential at right angles.

16. The solution of hydrodynamical problems is much simplified by the use of the velocity potential (whenever one exists),

since it enables us to express the velocities in terms of a single function ϕ. But when a velocity potential does not exist, this cannot in general be done, unless the motion either takes place in two dimensions, or is symmetrical with respect to an axis.

In the case of a liquid, if the motion takes place in planes parallel to the plane of xy, the equation of the lines of flow is

$$udy - vdx = 0 \quad\text{.......................... (22).}$$

The equation of continuity is

$$\frac{du}{dx} + \frac{dv}{dy} = 0,$$

which shows that the left-hand side of (22) is a perfect differential $d\psi$, whence

$$u = \frac{d\psi}{dy}, \quad v = -\frac{d\psi}{dx} \quad\text{.................. (23).}$$

The function ψ is called Earnshaw's current function.

When the motion takes place in planes passing through the axis of z, the equation of the lines of flow may be written

$$\varpi(wd\varpi - udz) = 0 \quad\text{..................... (24).}$$

The equation of continuity is

$$\frac{d(\varpi u)}{d\varpi} + \varpi\frac{dw}{dz} = 0,$$

which shows that the left-hand side of (24) is a perfect differential $d\psi$, whence

$$w = \frac{1}{\varpi}\frac{d\psi}{d\varpi}, \quad u = -\frac{1}{\varpi}\frac{d\psi}{dz} \quad\text{.............. (25),}$$

where ψ is Stokes' current function.

17. The existence of a velocity potential function involves the conditions that each of the three quantities,

$$dw/dy - dv/dz, \quad du/dz - dw/dx, \quad dv/dx - du/dy,$$

should be everywhere zero; when such is not the case we shall denote the above quantities by 2ξ, 2η, 2ζ. The quantities ξ, η, ζ, for reasons which will be explained in the following chapter, are called the components of molecular rotation. They evidently satisfy the equation

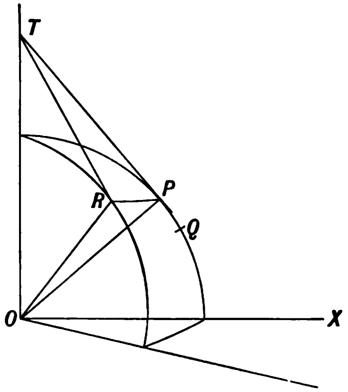
$$\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz} = 0 \quad\text{..................... (26).}$$

Formulae of Transformation[1].

18. The equations connecting the components of molecular relation with the velocities are,

$$2\xi = \frac{dw}{dy} - \frac{dv}{dz}, \quad 2\eta = \frac{du}{dz} - \frac{dw}{dx}, \quad 2\zeta = \frac{dv}{dx} - \frac{du}{dy} \quad \text{.......(27)}.$$

In order to obtain the equivalent equations when polar coordinates are employed, let r, θ, ϕ be the coordinates of P, and let u, v, w and $u + \delta u$, $v + \delta v$, $w + \delta w$ be the velocities at the points r, θ, ϕ and $r + \delta r$, $\theta + \delta\theta$, $\phi + \delta\phi$ respectively, measured in the directions in which these quantities increase; also let $u + \delta u'$, $v + \delta v'$, $w + \delta w'$ be the velocities at the last mentioned point parallel to the directions of u, v, w.



Let us choose the axes of x, y, z so as to coincide with the directions of r, θ, and ϕ respectively, then

$$dx = dr, \quad dy = rd\theta, \quad dz = r \sin\theta d\phi,$$

and therefore we at once obtain

$$\frac{du'}{dx} = \frac{du}{dr}, \quad \frac{dv'}{dx} = \frac{dv}{dr}, \quad \frac{dw'}{dx} = \frac{dw}{dr} \quad \text{................(28)}.$$

Let Q be a point whose coordinates are r, $\theta + \delta\theta$, ϕ; then

$$\frac{du'}{dy} = \frac{\left(u + \frac{du}{d\theta}\delta\theta\right)\cos\delta\theta - \left(v + \frac{dv}{d\theta}\delta\theta\right)\sin\delta\theta - u}{r\delta\theta}$$

$$= \frac{1}{r}\frac{du}{d\theta} - \frac{v}{r} \quad \text{..(29)},$$

$$\frac{dv'}{dy} = \frac{\left(v + \frac{dv}{d\theta}d\theta\right)\cos\delta\theta + \left(u + \frac{du}{d\theta}\delta\theta\right)\sin\delta\theta - v}{r\delta\theta}$$

$$= \frac{1}{r}\frac{dv}{d\theta} + \frac{u}{r} \quad \text{..(30)},$$

$$\frac{dw'}{dy} = \frac{1}{r}\frac{dw}{d\theta} \quad \text{..(31)}.$$

[1] Besant, *Mess. of Math.*, vol. XI. p. 63.

Let R be a point whose coordinates are $r, \theta, \phi+\delta\phi$; and let $POR=\delta\chi$, $PTR=\delta\chi'$; then

$$\delta\chi = \sin\theta\delta\phi,\ \delta\chi' = \cos\theta\delta\phi.$$

Hence

$$\frac{du'}{dz} = \frac{\left(u+\frac{du}{d\phi}\delta\phi\right)\cos\delta\chi - \left(w+\frac{dw}{d\phi}\delta\phi\right)\sin\delta\chi - u}{r\sin\theta\delta\phi}$$

$$= \frac{1}{r\sin\theta}\frac{du}{d\phi} - \frac{w}{r} \quad\text{.........(32)},$$

$$\frac{dv'}{dz} = \frac{\left(v+\frac{dv}{d\phi}\delta\phi\right)\cos\delta\chi' - \left(w+\frac{dw}{d\phi}\delta\phi\right)\sin\delta\chi' - v}{r\sin\theta\delta\phi}$$

$$= \frac{1}{r\sin\theta}\frac{dv}{d\phi} - \frac{w}{r}\cot\theta \quad\text{.........(33)},$$

$$\frac{dw'}{dz} = \frac{\left(w+\frac{dw}{d\phi}\delta\phi\right)\cos\delta\phi + \left(u+\frac{du}{d\phi}\delta\phi\right)\sin\delta\chi + \left(v+\frac{dv}{d\phi}\delta\phi\right)\sin\delta\chi' - w}{r\sin\theta\delta\theta}$$

$$= \frac{1}{r\sin\theta}\frac{dw}{d\phi} + \frac{u}{r} + \frac{v}{r}\cot\theta \quad\text{.........(34)}.$$

Hence

$$\left.\begin{aligned} 2\xi &= \frac{dw'}{dy} - \frac{dv'}{dz} = \frac{1}{r}\frac{dw}{d\theta} + \frac{w}{r}\cot\theta - \frac{1}{r\sin\theta}\frac{dv}{d\phi} \\ 2\eta &= \frac{du'}{dz} - \frac{dw'}{dx} = \frac{1}{r\sin\theta}\frac{du}{d\phi} - \frac{dw}{dr} - \frac{w}{r} \\ 2\zeta &= \frac{dv'}{dx} - \frac{du'}{dy} = \frac{dv}{dr} + \frac{v}{r} - \frac{1}{r}\frac{du}{d\theta} \end{aligned}\right\}\text{......(35)}.$$

19. If cylindrical coordinates ϖ, θ, z are employed; let u, v, w and $u+\delta u, v+\delta v, w+\delta w$ be the velocities at the points ϖ, θ, z and $\varpi+\delta\varpi, \theta+\delta\theta, z+\delta z$ respectively; and let $u+du'$, $v+dv'$ be the velocities at the last mentioned point parallel to u and v. Then

$$dx = d\varpi,\ dy = \varpi d\theta,$$

and

$$\frac{du'}{dx} = \frac{du}{d\varpi},\ \frac{dv'}{dx} = \frac{dv'}{d\varpi},\ \frac{dw'}{dx} = \frac{dw}{d\varpi} \quad\text{.........(36)},$$

also

$$\frac{du'}{dy} = \frac{\left(u+\frac{du}{d\theta}\delta\theta\right)\cos\delta\theta - \left(v+\frac{dv}{d\theta}\delta\theta\right)\sin\delta\theta - u}{\varpi\delta\theta}$$

$$= \frac{1}{\varpi}\frac{du}{d\theta} - \frac{v}{\varpi} \quad\text{.........(37)},$$

$$\frac{dv'}{dy}=\frac{\left(v+\dfrac{dv}{d\theta}\delta\theta\right)\cos\delta\theta+\left(u+\dfrac{du}{d\theta}\delta\theta\right)\sin\delta\theta-u}{\varpi\delta\theta}$$

$$=\frac{1}{\varpi}\frac{dv}{d\theta}+\frac{u}{\varpi}\quad\text{...(38)},$$

$$\frac{dw}{dy}=\frac{dw}{\varpi d\theta}\quad\text{...(39)},$$

and

$$\frac{du'}{dz}=\frac{du}{dz},\quad\frac{dv'}{dz}=\frac{dv}{dz}\quad\text{........................(40)}.$$

Therefore

$$\left.\begin{aligned}2\xi&=\frac{1}{\varpi}\frac{dw}{d\theta}-\frac{dv}{dz}\\2\eta&=\frac{du}{dz}-\frac{dw}{d\varpi}\\2\zeta&=\frac{dv}{d\varpi}+\frac{v}{\varpi}-\frac{1}{\varpi}\frac{du}{d\theta}\end{aligned}\right\}\quad\text{..................(41)}.$$

EXAMPLES.

1. Find the equation of continuity in a form suitable for air in a tube, and prove that if the density be $f(at-x)$ when t is the time and x the distance from one end of a uniform tube, the velocity is

$$\frac{af(at-x)+(V-a)f(at)}{f(at-x)},$$

where

where V is the velocity at that end of the tube.

2. If the motion of a liquid be in two dimensions, prove that if at any instant the velocity be everywhere the same in magnitude, it is so in direction.

3. If every particle of a fluid move in the surface of a sphere, prove that the equation of continuity is

$$\frac{d\rho}{dt}\cos\theta+\frac{d}{d\theta}(\rho\omega\cos\theta)+\frac{d}{d\phi}(\rho\omega'\cos\theta)=0,$$

where ρ is the density, θ and ϕ the latitude and longitude of any element, and ω, ω' the angular velocities of the element in latitude and longitude respectively.

4. In the last example prove that if the motion is irrotational the velocity potential is equal to

$$f(\log\tan\tfrac{1}{2}\theta+\iota\phi)+F(\log\tan\tfrac{1}{2}\theta-\iota\phi),$$

where $\iota=\sqrt{-1}$ and f and F are arbitrary functions.

5. An infinite mass of liquid is bounded by the plane zx, on which are small corrugations given by $y=\phi(x)$. The velocity of the liquid at an infinite distance from the plane is parallel to x and equal to V. Prove that the velocity potential is

$$Vx+\frac{V}{\pi}\int_{-\infty}^{\infty}\frac{(x-\lambda)\,\phi(\lambda)\,d\lambda}{y^2+(x-\lambda)^2}.$$

6. In the general motion of a fluid, prove that if F is the normal acceleration at any point on a closed surface described in a fluid, θ the expansion, ω the molecular rotation, and Σ the strain invariant

$fg+gh+hf-a^2-b^2-c^2$, where $f=du/dx$, $2a=dw/dy+dv/dz$,

then $$\iint F\,dS=\iiint\left(\frac{\partial\theta}{\partial t}+\theta^2+2\omega^2-2\Sigma\right)dx\,dy\,dz.$$

7. Fluid is moving in a fine tube of variable section κ, prove that the equation of continuity is

$$\frac{d}{dt}(\kappa\rho)+\frac{d}{ds}(\kappa\rho v)=0,$$

where v is the velocity at the point s.

8. If $F(x, y, z, t)$ is the equation of a moving surface the velocity of the surface normal to itself is

$$-\frac{1}{R}\frac{dF}{dt},\ \text{where } R^2=(dF/dx)^2+(dF/dy)^2+(dF/dz)^2.$$

Hence deduce equation (19).

9. If x, y and z are given functions of a, b, c and t, where a, b and c are constants for any particular element of fluid, and if u, v and w are the values of $\dot{x}$, $\dot{y}$, $\dot{z}$ when a, b, c are eliminated, prove analytically that

$$\frac{d^2x}{dt^2}=\frac{du}{dt}+u\frac{du}{dx}+v\frac{du}{dy}+w\frac{du}{dz}.$$

10. Liquid which is moving irrotationally in three dimensions is bounded by the ellipsoid $(x/a)^2+(y/b)^2+(z/c)^2=1$, where

a, b, c are functions of the time, such that the volume of the ellipsoid remains constant. Prove that if the ellipsoid is rotating with angular velocities ω_1, ω_2, ω_3 about its principal axes, and u, v, w are the component velocities of the liquid parallel to the principal axes, the equation of continuity and the boundary conditions are satisfied if

$$u = \frac{\dot{a}x}{a} + \frac{\omega_3 (a^2 - b^2) y}{a^2 + b^2} + \frac{\omega_2 (c^2 - a^2) z}{c^2 + a^2},$$

with similar expressions for v and w.

11. If the lines of flow of a fluid lie on the surfaces of coaxial cones having the same vertex, prove that the equation of continuity is $$r\frac{d\rho}{dt} + r\frac{d}{dr}(u\rho) + 2\rho u + \operatorname{cosec}\theta \frac{d}{d\phi}(\rho v) = 0.$$

12. Show that

$$x^2/(akt^2)^2 + kt^2\{(y/b)^2 + (z/c)^2\} = 1$$

is a possible form of the bounding surface at time t of a liquid.

13. The position of a point in a plane is determined by the length r of the tangent from it to a fixed circle of radius a, and the inclination θ of the tangent to a fixed line. Show that the equation of continuity for a liquid moving irrotationally in the plane will be

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} + \frac{1}{r^2}\frac{d^2\phi}{d\theta^2} + \frac{a^2}{r^2}\left(\frac{d^2\phi}{dr^2} - \frac{1}{r}\frac{d\phi}{dr}\right) + \frac{a}{r^2}\left(2\frac{d^2\phi}{drd\theta} - \frac{1}{r}\frac{d\phi}{d\theta}\right) = 0.$$

Hence indicate a method of finding the motion of a liquid in the developable surface whose edge of regression is a right helix, pointing out any peculiarities of the motion.

14. If the velocity potential of a liquid is of the form $\phi = f(\varpi) F(\theta) \chi(z)$, where ϖ, θ, z are cylindrical coordinates, prove that the equation of continuity is satisfied if f, F, χ satisfy the three equations

$$\varpi^2\frac{d^2f}{d\varpi^2} + \varpi\frac{df}{d\varpi} + (\kappa^2\varpi^2 - n^2)f = 0,\ \frac{d^2F}{d\theta^2} + n^2F = 0,\ \frac{d^2\chi}{dz^2} - \kappa^2\chi = 0,$$

where n and κ are constants; and hence show that

$$\phi = \Sigma A \cosh\kappa(z - c)\cos n(\theta - \alpha)\int_0^\pi \cos(\kappa r\sin\omega - n\omega)\, d\omega.$$

15. In the motion of a liquid in two dimensions, the velocity at any point is given by two components v, v' along the directions which pass through two fixed points distant a from one another. Show that the equation of continuity is

$$\frac{dv}{dr} + \frac{dv'}{dr} + \frac{r^2 + r'^2 - a^2}{2rr'}\left(\frac{dv}{dr'} + \frac{dv'}{dr}\right) + \frac{v}{r} + \frac{v'}{r'} = 0,$$

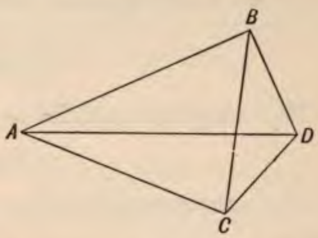
where r, r' are the distances of any point of the liquid from the fixed points.

CHAPTER II.

ON THE GENERAL EQUATIONS OF MOTION OF A PERFECT FLUID.

20. It was stated in the preceding chapter, that the pressure at every point of a perfect fluid is equal in all directions, whether the fluid be at rest or in motion. It will now be shown that this property is the consequence of such a fluid being incapable of offering resistance to a tangential stress.

Let $ABCD$ be a small tetrahedron of fluid, and let p, p' be the pressures per unit of area upon the faces ABC and BCD.



By D'Alembert's Principle, the reversed effective forces and the impressed forces which act upon the volume of fluid, together with the pressures upon its faces, constitute a system in statical equilibrium. The first two vary as the volume, and the last vary as the areas of the faces of the tetrahedron; and therefore if the tetrahedron be made to diminish indefinitely, the former will vanish in comparison with the latter. Hence the tetrahedron will ultimately be in equilibrium under the action of the pressures upon its faces.

Resolve the pressures upon the faces ABC and BCD parallel to AD. Since the projections of the two faces upon a plane perpendicular to AD are equal, the conditions of equilibrium require that $p = p'$, which proves the proposition[1].

[1] This proposition is true even in the case of viscous fluids, provided they are at rest.

The Equations of Motion[1].

21. Let X, Y, Z be the components per unit of mass of the impressed forces which act on the fluid; ρ its density, and q its resultant velocity. Describe any imaginary closed surface S in the fluid, and let ϵ be the angle which the direction of q makes with the normal to S drawn outwards.

The rate at which momentum flows into S, parallel to x, together with the rate of increase of the component of momentum parallel to x, of the fluid contained within S, must be equal to the component parallel to x of the impressed forces which act on the fluid within S, together with the component parallel to x of the pressure upon the boundary of S.

The rate at which momentum flows into S, parallel to x, is

$$\iint \rho q^2 l \cos \epsilon dS = \iint \rho u (lu + mv + nw) dS$$
$$= \iiint \left\{ \frac{d(\rho u^2)}{dx} + \frac{d(\rho uv)}{dy} + \frac{d(\rho uw)}{dz} \right\} dx dy dz$$

by § 7.

The rate of increase of the component of momentum parallel to x of the fluid contained within S

$$= \iiint \frac{d}{dt}(\rho u)\, dx dy dz.$$

The component parallel to x of the impressed forces

$$= \iiint \rho X \, dx dy dz.$$

The component parallel to x of the pressure upon the boundary of S, is

$$- \iint p l dS = - \iiint \frac{dp}{dx} dx dy dz.$$

Whence

$$\iiint \left(\rho X - \frac{dp}{dx} \right) dx dy dz = \iiint \left\{ \frac{d(\rho u)}{dt} + \frac{d(\rho u^2)}{dx} + \frac{d(\rho uv)}{dy} + \frac{d(\rho uw)}{dz} \right\} dx dy dz,$$

which requires that

$$\rho X - \frac{dp}{dx} = \frac{d(\rho u)}{dt} + \frac{d(\rho u^2)}{dx} + \frac{d(\rho uv)}{dy} + \frac{d(\rho uw)}{dz}.$$

[1] This method of obtaining the equations of motion is due to Prof. Greenhill. See *Encyc. Brit., Art. Hydrodynamics*.

Taking account of the equation of continuity § 8, (10) the right hand side of the last equation becomes equal to $\rho\,\partial u/\partial t$, whence

$$\rho X - \frac{dp}{dx} = \rho\frac{\partial u}{\partial t}.$$

Two other symmetrical equations can be obtained, by considering the rate of change of momentum parallel to the other two axes, whence the equations of motion are

$$\left.\begin{aligned} X - \frac{1}{\rho}\frac{dp}{dx} &= \frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz} \\ Y - \frac{1}{\rho}\frac{dp}{dy} &= \frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz} \\ Z - \frac{1}{\rho}\frac{dp}{dz} &= \frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz} \end{aligned}\right\} \quad \text{.........(1).}$$

These equations together with the equation of continuity furnish four relations between the five quantities u, v, w, p, ρ.

22. If the fluid be an incompressible liquid, ρ is constant, and the above mentioned equations together with the boundary conditions are sufficient to determine the motion; but in the case of a gas another equation is required, which is furnished by means of a relation which exists between p and ρ.

When the motion of the gas is such that the temperature remains constant, we have by Boyle's Law the equation

$$p = k\rho \quad \text{..............................(2),}$$

where k is a constant.

But when the motion is such as to cause a sudden compression or dilatation, an increase or decrease of temperature will be produced; and if it is assumed (as is the case with sound waves), that the compression is so sudden that loss or gain of heat by radiation may be neglected, the required relation is

$$p = k\rho^\gamma \quad \text{..............................(3),}$$

where γ is the ratio of the specific heat at constant pressure to the specific heat at constant volume[1]. This quantity for all gases has the approximately constant value 1·408.

23. The expressions on the right hand of (1) are the expressions for the component accelerations of an element of fluid; it therefore follows that if F and f be the component force and

[1] This equation will be proved in the Appendix.

acceleration in any direction, and dp/dh be the space variation of the pressure, the equations of motion are of the form

$$F - \frac{1}{\rho}\frac{dp}{dh} = f.$$

Hence if the axes instead of being fixed are moving with angular velocities θ_1, θ_2, θ_3 about themselves, the equations of motion will be obtained by employing the expressions for the accelerations given in § 6, (6), and are therefore,

$$\left.\begin{aligned} X - \frac{1}{\rho}\frac{dp}{dx} &= \frac{du}{dt} + U\frac{du}{dx} + V\frac{du}{dy} + W\frac{du}{dz} - v\theta_3 + w\theta_2 \\ Y - \frac{1}{\rho}\frac{dp}{dy} &= \frac{dv}{dt} + U\frac{dv}{dx} + V\frac{dv}{dy} + W\frac{dv}{dz} - w\theta_1 + u\theta_3 \\ Z - \frac{1}{\rho}\frac{dp}{dz} &= \frac{dw}{dt} + U\frac{dw}{dx} + V\frac{dw}{dy} + W\frac{dw}{dz} - u\theta_2 + v\theta_1 \end{aligned}\right\} \ldots(4).$$

24. Let us now suppose that the forces arise from a conservative system whose potential is V. Since p is a function of ρ, we may put

$$Q = -\int\frac{dp}{\rho} - V,$$

and the left-hand sides of (1), will be respectively equal to dQ/dx, dQ/dy, dQ/dz. If therefore we eliminate Q by differentiating the second equation with respect to z and the third with respect to y, we shall obtain

$$\frac{\partial\xi}{\partial t} = \xi\frac{du}{dx} + \eta\frac{dv}{dx} + \zeta\frac{dw}{dx} - \xi\theta,$$

where ξ, η, ζ are the components of molecular rotation and $\theta = du/dx + dv/dy + dw/dz$. Eliminating θ by means of the equation of continuity $\partial\rho/\partial t + \rho\theta = 0$, and taking account of the two other equations which may be written down from symmetry, we shall obtain

$$\left.\begin{aligned} \frac{\partial}{\partial t}\left(\frac{\xi}{\rho}\right) &= \frac{\xi}{\rho}\frac{du}{dx} + \frac{\eta}{\rho}\frac{dv}{dx} + \frac{\zeta}{\rho}\frac{dw}{dx} \\ \frac{\partial}{\partial t}\left(\frac{\eta}{\rho}\right) &= \frac{\xi}{\rho}\frac{du}{dy} + \frac{\eta}{\rho}\frac{dv}{dy} + \frac{\zeta}{\rho}\frac{dw}{dy} \\ \frac{\partial}{\partial t}\left(\frac{\zeta}{\rho}\right) &= \frac{\xi}{\rho}\frac{du}{dz} + \frac{\eta}{\rho}\frac{dv}{dz} + \frac{\zeta}{\rho}\frac{dw}{dz} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots(5).$$

25. It was stated in Chapter I., that in many important problems the motion is such that a velocity potential exists.

The condition that such should be the case is, that ξ, η, ζ should each vanish. We shall now prove, that when the fluid is under the action of a conservative system of forces, a velocity potential will always exist whenever it exists at any particular instant.

Let us choose the particular instant at which a velocity potential exists as the origin of the time; then by hypotheses ξ, η, ζ vanish when $t = 0$; also the coefficients of these quantities in (5), will not become infinite at any point of the interior of the fluid; it will therefore be possible to determine a quantity L, which shall be a superior limit to the numerical values of these coefficients. Hence ξ, η, ζ cannot increase faster than if they satisfied the equations

$$\frac{\partial}{\partial t}\left(\frac{\xi}{\rho}\right) = \frac{L}{\rho}(\xi + \eta + \zeta), \text{ \&c. \&c.}$$

But if $\xi + \eta + \zeta = \Omega\rho$, we obtain by adding the above equations

$$\frac{\partial \Omega}{\partial t} = 3L\Omega,$$

whence

$$\Omega = Ae^{3Lt}.$$

Now $\Omega = 0$ when $t = 0$, therefore $A = 0$; and since Ω is the sum of three quantities each of which is essentially positive, it follows that ξ, η, ζ must always remain zero, if they are so at any particular instant. The above proof is due to Prof. Stokes[1].

26. There is, as was first shown by Prof. Stokes, an important physical distinction in the character of the motion which takes place, according as a velocity potential does or does not exist.

Conceive an indefinitely small spherical element of a fluid in motion to become suddenly solidified, and the fluid about it to be suddenly destroyed. By the instantaneous solidification velocities will be suddenly generated or destroyed in the different portions of the element, and a set of mutual impulsive forces will be called into action.

Let x, y, z be the coordinates of the centre of inertia G of the element at the instant of solidification, $x + x'$, $y + y'$, $z + z'$ those of any other point P in it; let u, v, w be the velocities of G along the three axes just before solidification, u' v', w' the velocities of P relative to G; also let $\bar{u}$, $\bar{v}$, $\bar{w}$ be the velocities of G, u_1, v_1, w_1 the relative velocities of P, and ξ, η, ζ the angular velocities just

[1] "On the friction of fluids in motion," Section II. *Trans. Camb. Phil. Soc.* vol. VIII.

after solidification. Since all the impulsive forces are internal, we have

$$\bar{u} = u, \quad \bar{v} = v, \quad \bar{w} = w.$$

We have also, by the principle of conservation of angular momentum, $\Sigma m \{y'(w_1 - w') - z'(v_1 - v')\} = 0$, &c.

m denoting an element of the mass of the element considered.

But $u_1 = \eta z' - \zeta y'$, and u' is ultimately equal to

$$\frac{du}{dx} x' + \frac{du}{dy} y' + \frac{du}{dz} z',$$

and similar expressions hold good for the other quantities. Substituting in the above equation, and observing that

$$\Sigma my'z' = \Sigma m'z'x' = \Sigma mx'y' = 0, \text{ and } \Sigma mx'^2 = \Sigma my'^2 = \Sigma mz'^2,$$

we have
$$\xi = \tfrac{1}{2}\left(\frac{dw}{dy} - \frac{dv}{dz}\right), \text{ \&c.}$$

We see then that an indefinitely small spherical element of the fluid if suddenly solidified and detached from the rest of the fluid will begin to move with a motion of translation alone, or a motion of translation combined with one rotation, according as $udx + vdy + wdz$ is, or is not, an exact differential, and in the latter case the angular velocities will be determined by the equations

$$2\xi = \frac{dw}{dy} - \frac{dv}{dz}, \quad 2\eta = \frac{du}{dz} - \frac{dw}{dx}, \quad 2\zeta = \frac{dv}{dx} - \frac{du}{dy}.$$

On account of the physical meaning of the quantities ξ, η, ζ, they are called the *components of molecular rotation*, and motion which is such that they do not vanish is called *rotational or vortex motion;* when they vanish, the motion is called *irrotational.*

In the foregoing investigations, it has been assumed that the pressure is a function of the density and also that the fluid is under the action of a conservative system of forces; it therefore follows that vortex motion cannot be produced, and consequently, if once set up, cannot be destroyed by such a system of forces. We shall presently show that the theorem is not true if the pressure is not a function of the density. If therefore by reason of any chemical action the pressure should cease to be a function of the density during any interval of time however short, vortex motion might be produced, or if in existence might be destroyed.

Lagrange's Equations.

27. In Lagrange's method the initial coordinates a, b, c and the time are the independent variables, hence the equations of motion are

$$\frac{dQ}{dx} = \dot{u}, \quad \frac{dQ}{dy} = \dot{v}, \quad \frac{dQ}{dz} = \dot{w}.$$

Multiplying the preceding equations by x_a, y_a, z_a, where the suffixes denote partial differentiation with respect to a, b, c, we obtain

$$\left.\begin{aligned} Q_a &= \dot{u}x_a + \dot{v}y_a + \dot{w}z_a \\ Q_b &= \dot{u}x_b + \dot{v}y_b + \dot{w}z_b \\ Q_c &= \dot{u}x_c + \dot{v}y_c + \dot{w}z_c \end{aligned}\right\} \quad \text{..................(6).}$$

These equations together with the equation of continuity $\rho J = \rho_0$, are Lagrange's hydrodynamical equations of motion.

Weber's Transformation.

28. Integrating the right hand side of the first of (6) between the limits t and 0, the first term becomes

$$\begin{aligned} \int_0^t \dot{u}x_a\,dt &= \int_0^t \ddot{x}x_a\,dt = (\dot{x}x_a)_0^t - \int_0^t \dot{x}\dot{x}_a\,dt \\ &= ux_a - u_0 - \frac{1}{2}\frac{d}{da}\int_0^t u^2 dt, \end{aligned}$$

where u_0 is the initial value of u. If we treat each of the other two terms in a similar manner and put

$$\chi = \int_0^t (Q + \tfrac{1}{2}q^2)\,dt,$$

where q is the resultant velocity of the liquid, we obtain

$$\left.\begin{aligned} ux_a + vy_a + wz_a - u_0 &= \frac{d\chi}{da} \\ ux_b + vy_b + wz_b - v_0 &= \frac{d\chi}{db} \\ ux_c + vy_c + wz_c - w_0 &= \frac{d\chi}{dc} \end{aligned}\right\} \quad \text{.................. (7).}$$

These equations together with the equation of continuity and $d\chi/dt = Q + \frac{1}{2}q^2$, give five equations for determining x, y, z, p, χ; ρ being supposed to have been eliminated by means of (2) or (3).

The above equations may be expressed in a different form, for multiplying by dJ/dx_a, dJ/dx_b, dJ/dx_c and adding, we obtain

$$u = \frac{1}{J}\left(u_0\frac{dJ}{dx_a} + v_0\frac{dJ}{dx_b} + w_0\frac{dJ}{dx_c}\right) + \frac{d\chi}{dx} \quad \text{...........(8),}$$

with two similar equations.

29. Multiply (7) by da, db, dc and add, and we obtain

$$udx + vdy + wdz - u_0da - v_0db - w_0dc = d\chi \quad \text{.........(9)}.$$

If at any particular instant which we shall choose for the origin of the time a velocity potential exists, $u_0da + v_0db + w_0dc$ will be a perfect differential; hence if p be a function of ρ, $d\chi$ will also be a perfect differential, which proves that if a velocity potential once exists, it will always exist; but if p is not a function of ρ we cannot put $Q = -V - \int\rho^{-1}\,dp$, but must write

$$\frac{d}{da}\int_0^t (\tfrac{1}{2}q^2 - V)\,dt - \int_0^t \frac{1}{\rho}\frac{dp}{da}\,dt$$

for $d\chi/da$, in which case the right hand side of (9) becomes

$$d\int_0^t (\tfrac{1}{2}q^2 - V)\,dt - \int_0^t \left(\frac{dp}{\rho}\right) dt$$

where d denotes space differentiation. The right hand side of (9) is no longer a perfect differential; hence $udx + vdy + wdz$ is *not* a perfect differential.

If therefore the pressure be *not* a function of the density, vortex motion can be generated or destroyed in a perfect fluid moving under the action of natural forces.

Cauchy's Integrals.

30. Eliminating Q from the last two of (6), we obtain

$$\dot{u}_b x_c - \dot{u}_c x_b + \dot{v}_b y_c - \dot{v}_c y_b + \dot{w}_b z_c - \dot{w}_c z_b = 0.$$

Integrate this equation with respect to t, and let u_0, v_0, w_0 be the initial values of u, v, w; then

$$u_b x_c - u_c x_b + v_b y_c - v_c y_b + w_b z_c - w_c z_b = \frac{dw_0}{db} - \frac{dv_0}{dc}.$$

But
$$u_a = \frac{du}{dx}x_a + \frac{du}{dy}y_a + \frac{du}{dz}z_a, \text{ \&c. \&c.}$$

Substituting these values of u_a, u_b, &c., we obtain the equations

$$\xi\frac{dJ}{dx_a} + \eta\frac{dJ}{dy_a} + \zeta\frac{dJ}{dz_a} = \xi_0,$$

$$\xi\frac{dJ}{dx_b} + \eta\frac{dJ}{dy_b} + \zeta\frac{dJ}{dz_b} = \eta_0,$$

$$\xi\frac{dJ}{dx_c} + \eta\frac{dJ}{dy_c} + \zeta\frac{dJ}{dz_c} = \zeta_0.$$

Multiplying these equations by x_a, x_b, x_c and adding, and remembering that $J\rho = \rho_0$, we obtain

$$\left.\begin{aligned}\frac{\rho_0\xi}{\rho} &= \xi_0 x_a + \eta_0 x_b + \zeta_0 x_c\\ \frac{\rho_0\eta}{\rho} &= \xi_0 y_a + \eta_0 y_b + \zeta_0 y_c\\ \frac{\rho_0\zeta}{\rho} &= \xi_0 z_a + \eta_0 z_b + \zeta_0 z_c\end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots(10).$$

These equations show that ξ, η, ζ are always zero, if they are initially so.

31. The equations of motion can be integrated whenever a force and a velocity potential exist; for putting

$$Q = -\int\frac{dp}{\rho} - V,$$

and multiplying (1) by dx, dy, dz respectively and adding, we obtain

$$dQ = \frac{\partial u}{\partial t}dx + \frac{\partial v}{\partial t}dy + \frac{\partial w}{\partial t}dz.$$

Now in the present case

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{du}{dt} + u\frac{du}{dx} + v\frac{dv}{dx} + w\frac{dw}{dx}\\ &= \frac{d}{dx}\left(\frac{d\phi}{dt} + \tfrac{1}{2}q^2\right),\end{aligned}$$

where q is the resultant velocity. Integrating, we obtain

$$\int\frac{dp}{\rho} + V + \frac{d\phi}{dt} + \tfrac{1}{2}q^2 = F(t)\ldots\ldots\ \ldots\ldots\ldots(11),$$

where F is an arbitrary function.

32. DEF. A vortex line is a line whose direction coincides with the direction of the instantaneous axis of molecular rotation.

The differential equations of a vortex line are thus

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\zeta}.$$

Clebsch's Transformation[1].

33. When a velocity potential does not exist, a first integral of the general equations of motion can be obtained by means of a method which depends upon the analytical theorem, that if u, v, w are any given functions of x, y, z it is always possible to determine three quantities ϕ, λ, χ, such that

$$udx + vdy + wdz = d\phi + \lambda d\chi \quad\text{.................(12).}$$

In order to prove the theorem, let u', v', w', ϕ be four quantities, such that

$$u = u' + \phi_x, \quad v = v' + \phi_y, \quad w = w' + \phi_z.$$

These equations involve three relations between the four quantities u', v', w', ϕ and are therefore insufficient to determine them as functions of u, v, w; we may therefore assume any relation between u', v', w' which may be convenient. Let us therefore suppose that

$$u'(w'_y - v'_z) + v'(u'_z - w'_x) + w'(v'_x - u'_y) = 0.$$

This is the condition that $u'dx + v'dy + w'dz$ should have an integrating factor, we may therefore put this quantity equal to $\lambda d\chi$ which proves the proposition. It therefore follows that,

$$u = \frac{d\phi}{dx} + \lambda\frac{d\chi}{dx}, \quad v = \frac{d\phi}{dy} + \lambda\frac{d\chi}{dy}, \quad w = \frac{d\phi}{dz} + \lambda\frac{d\chi}{dz} \quad\text{......(13).}$$

The components of molecular rotation are given by the equations

$$\left.\begin{aligned} 2\xi &= \lambda_y\chi_z - \lambda_z\chi_y \\ 2\eta &= \lambda_z\chi_x - \lambda_x\chi_z \\ 2\zeta &= \lambda_x\chi_y - \lambda_y\chi_x \end{aligned}\right\} \quad\text{.....................(14).}$$

The form of these equations shows that the vortex lines are the intersections of the surfaces $\lambda = \text{const.}$, $\chi = \text{const.}$

Now $$\frac{du}{dt} = \frac{d}{dx}\left(\frac{d\phi}{dt} + \lambda\frac{d\chi}{dt}\right) + \frac{d\lambda}{dt}\chi_x - \frac{d\chi}{dt}\lambda_x.$$

Therefore

$$\frac{\partial u}{\partial t} = \frac{d}{dx}\left(\frac{d\phi}{dt} + \lambda\frac{d\chi}{dt}\right) + u\frac{du}{dx} + v\frac{dv}{dx} + w\frac{dw}{dx}$$
$$+ \frac{\partial\lambda}{\partial t}\chi_x - \frac{\partial\chi}{\partial t}\lambda_x \quad\text{...............(15).}$$

[1] *Crelle*, vol. LVI. p. 1. See also Hill, *Quart. Journ.* vol. XVII; *Trans. Camb. Phil. Soc.* vol. XIV. p. 1; *Phil. Trans.* 1884, p. 363; *Proc. Lond. Math. Soc.* vol. XVI. p. 171.

Putting
$$H = -Q + \frac{d\phi}{dt} + \lambda\frac{d\chi}{dt} + \tfrac{1}{2}q^2 \quad\text{...............(16),}$$
and substituting the values of $\partial u/\partial t$ and dQ/dx from (15) and (16) in (1), we obtain
$$\frac{dH}{dx} + \frac{\partial\lambda}{\partial t}\chi_x - \frac{\partial\chi}{\partial t}\lambda_x = 0 \quad\text{...............(17),}$$
with two similar equations.

Multiplying by ξ, η, ζ and adding, we obtain
$$\xi\frac{dH}{dx} + \eta\frac{dH}{dy} + \xi\frac{dH}{dz} = 0 \quad\text{...............(18).}$$

If ds be an element of a vortex line, and ω be the resultant molecular rotation, the operator is equal to $\omega d/ds$, whence integrating along a vortex line, we obtain
$$\int\frac{dp}{\rho} + V + \frac{d\phi}{dt} + \lambda\frac{d\chi}{dt} + \tfrac{1}{2}q^2 = F(t, \lambda, \chi)\quad\text{.........(19).}$$

Writing for a moment $P = \partial\lambda/\partial t$, $R = \partial\chi/\partial t$ and eliminating H from (17), we obtain
$$\begin{aligned} P_y\chi_x - R_y\lambda_x - P_x\chi_y + R_x\lambda_y &= 0\\ P_z\chi_y - R_z\lambda_y - P_y\chi_z + R_y\lambda_z &= 0\\ P_x\chi_z - R_x\lambda_z - P_z\chi_x + R_z\lambda_x &= 0. \end{aligned}$$

Multiplying these equations in order by λ_z, λ_x, λ_y and adding and taking account of (14), we obtain
$$\xi P_x + \eta P_y + \zeta P_z = 0 \quad\text{..................(20).}$$

If x, y, z be *any* point on the surface $\lambda = A$, where A is an absolute constant, and if ξ/ω, η/ω, ζ/ω be the direction cosines of the vortex line at this point; equations (14) and (20) show that this vortex line lies on the surfaces $\lambda = A$ and $\lambda + \partial\lambda/\partial t \,.\, dt = A$, which is impossible unless $\partial\lambda/\partial t = 0$. Similarly $\partial\chi/\partial t = 0$; whence the surfaces λ and χ and therefore the vortex lines are always composed of the same elements of fluid. This important theorem was first established by Helmholtz[1].

Hence it follows from (17) that H_x, H_y, H_z are each equal to zero, and therefore H is a function of the time alone; whence the pressure is determined by the equation
$$\int\frac{dp}{\rho} + V + \frac{d\phi}{dt} + \lambda\frac{d\chi}{dt} + \tfrac{1}{2}q^2 = F(t)\quad\text{............(21).}$$

[1] *Crelle*, vol. LV. and *Phil. Mag.* (4) vol. XXXIII. p. 485.

34. We can now show that in the case of a liquid, the integral

$$\iiiint \left(\frac{p}{\rho} + V\right) dt\,dx\,dy\,dz \qquad \text{.................(22)},$$

is a maximum or minimum, where the value of $p/\rho + V$ or $-Q$ is given by (21), and the time remains invariable.

For $$\delta Q = u\delta u + v\delta v + w\delta w + \frac{d\delta\phi}{dt} + \frac{d\chi}{dt}\delta\lambda + \lambda\frac{d\delta\chi}{dt},$$

and $$\delta u = \frac{d\delta\phi}{dx} + \frac{d\chi}{dx}\delta\lambda + \lambda\frac{d\delta\chi}{dx}.$$

Therefore

$$\iiiint u\delta u\, dt\,dx\,dy\,dz = \iiint u\,(\delta\phi + \lambda\delta\chi)\,dt\,dy\,dz$$
$$+ \iiiint \left\{u\chi_x\delta\lambda - \frac{d}{dx}(\lambda u)\,\delta\chi - \frac{du}{dx}\delta\phi\right\} dt\,dx\,dy\,dz.$$

Omitting the triple integrals which refer to the boundary we see that the first three terms of δQ give rise to the terms

$$\iiiint \{(u\chi_x + v\chi_y + w\chi_z)\,\delta\lambda - (u\lambda_x + v\lambda_y + w\lambda_z)\,\delta\chi$$
$$- \theta\,(\delta\phi + \lambda\delta\chi)\}\, dt\,dx\,dy\,dz,$$

which

$$= \iiiint \left\{\left(\frac{\partial\chi}{\partial t} - \frac{d\chi}{dt}\right)\delta\lambda - \left(\frac{\partial\lambda}{\partial t} - \frac{d\lambda}{dt}\right)\delta\chi - \theta\,(\delta\phi + \lambda\delta\chi)\right\} dt\,dx\,dy\,dz,$$

where $$\theta = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}.$$

Also the last three terms of δQ (omitting triple integrals) give rise to

$$\iiiint \left\{\frac{d\chi}{dt}\delta\lambda - \frac{d\lambda}{dt}\delta\chi\right\} dt\,dx\,dy\,dz.$$

Whence

$$\iiiint \delta Q\, dt\,dx\,dy\,dz = \iiiint \left\{\frac{\partial\chi}{\partial t}\delta\lambda - \frac{\partial\lambda}{\partial t}\delta\chi - \theta\,(\delta\phi + \lambda\delta\chi)\right\} dt\,dx\,dy\,dz$$
$$+ \text{ triple integrals.}$$

In order that the quadruple integral should vanish, we must have $\theta = 0$, $\partial\chi/\partial t = 0$, $\partial\lambda/\partial t = 0$, which by virtue of the equation of continuity and § 33 is obviously the case.

On the Application of the Principles of Energy and Least Action.

35. Let S be any imaginary closed surface, which is fixed in the ~~fluid~~ liquid. The work done during a small interval δt upon the liquid contained within S, by the impressed forces which act upon its mass, together with the work done by the pressure upon the boundary of S, must be equal to the increase during the interval δt of the kinetic energy of the liquid contained within S, together with the kinetic energy which, during the same interval, flows into S across the boundary.

The work done by the impressed forces

$$= -\iiint \rho \left(u\frac{dV}{dx} + v\frac{dV}{dy} + w\frac{dV}{dz}\right) \delta t\, dx dy dz.$$

The work done by the pressure upon the boundary

$$= -\iint p\,(lu + mv + nw)\,\delta t\, dS$$

$$= -\iiint \left(u\frac{dp}{dx} + v\frac{dp}{dy} + w\frac{dp}{dz}\right) \delta t\, dx dy dz,$$

by § 7. Hence the total work done

$$= \iiint \rho \left(u\frac{dQ}{dx} + v\frac{dQ}{dy} + w\frac{dQ}{dz}\right) \delta t\, dx dy dz.$$

Let T be the kinetic energy per unit of mass, so that

$$T = \tfrac{1}{2}(u^2 + v^2 + w^2).$$

The increase in the kinetic energy of the liquid contained within S

$$= \iiint \frac{d(T\rho)}{dt}\,\delta t\, dx dy dz.$$

The amount of kinetic energy which flows into S

$$= \iint \rho T\,(lu + mv + nw)\,\delta t\, dS$$

$$= \iiint \left\{\frac{d}{dx}(\rho u T) + \frac{d}{dy}(\rho v T) + \frac{d}{dz}(\rho w T)\right\} \delta t\, dx dy dz.$$

Taking account of the equation of continuity § 9 (10) the total increase in the kinetic energy

$$= \iiint \rho\,\frac{\partial T}{\partial t}\,\delta t\, dx dy dz.$$

Whence $\iiint \rho \left(\frac{\partial T}{\partial t} - u\frac{dQ}{dx} + v\frac{dQ}{dy} - w\frac{dQ}{dz}\right) \delta t\, dx\, dy\, dz = 0$

which requires that

$$\frac{\partial T}{\partial t} = u\frac{dQ}{dx} + v\frac{dQ}{dy} + w\frac{dQ}{dz} \qquad\qquad (23).$$

If we substitute the values of u, v, w from (13), we find that

$$\frac{dT}{dt} = \left(u\frac{d}{dx} + v\frac{d}{dy} + w\frac{d}{dz}\right)\left(\frac{d\phi}{dt} + \lambda\frac{d\chi}{dt}\right)$$
$$+ (u\chi_x + v\chi_y + w\chi_z)\frac{d\lambda}{dt} - (u\lambda_x + v\lambda_y + w\lambda_z)\frac{d\chi}{dt}.$$

The last two terms vanish by § 33, whence (23) becomes

$$\left(u\frac{d}{dx} + v\frac{d}{dy} + w\frac{d}{dz}\right)\left(Q - T - \frac{d\phi}{dt} - \lambda\frac{d\chi}{dt}\right) = 0.$$

Now if ds be an elementary arc of a stream line $u = q\,dx/ds$, &c., and the operator is therefore equal to $q\,d/ds$. Integrating along a stream line, and restoring the values of Q and T, we obtain

$$\frac{p}{\rho} + V + \tfrac{1}{2}q^2 + \frac{d\phi}{dt} + \lambda\frac{d\chi}{dt} = F(t).$$

36. The equations of motion may be deduced, as Mr Larmor has shown, by means of the Principle of Least Action combined with the Lagrangian method.

Let x, y, z be the coordinates at time t of an element of fluid whose initial coordinates are a, b, c; the Principle of Least Action requires that

$$\iiiint \{\tfrac{1}{2}\rho\,(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V\rho\}\, dt\, dx\, dy\, dz$$

should be a maximum or minimum subject to the condition that

$$J = \frac{d\,(x, y, z)}{d\,(a, b, c)} = \frac{\rho_0}{\rho},$$

where the time of the motion is constant.

Hence if λ represent an undetermined function of x, y, and z, we must have

$$\delta \iiiint \left\{\tfrac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V - \lambda\frac{d\,(x, y, z)}{d\,(a, b, c)}\right\} dt\, da\, db\, dc = 0.$$

Taking the variation of the first two terms, we obtain

$$\iiiint \left\{(\dot{x}\delta\dot{x} + \dot{y}\delta\dot{y} + \dot{z}\delta\dot{z}) - \left(\frac{dV}{dx}\delta x + \frac{dV}{dy}\delta y + \frac{dV}{dz}\delta z\right)\right\} dt\, da\, db\, dc.$$

Integrating by parts and omitting the triple integrals, this

$$=-\iiiint\left\{\left(\ddot{x}+\frac{dV}{dx}\right)\delta x+\left(\ddot{y}+\frac{dV}{dy}\right)\delta y+\left(\ddot{z}+\frac{dV}{dz}\right)\delta z\right\}dt\,da\,db\,dc.$$

If M_1, M_2, M_3 be the minors of dx/da, dx/db, dx/dc in J

$$\frac{d(\delta x, y, z)}{d(a, b, c)}=M_1\frac{d\delta x}{da}+M_2\frac{d\delta x}{db}+M_3\frac{d\delta x}{dc},$$

whence, omitting triple integrals,

$$\iiiint\lambda\frac{d(\delta x, y, z)}{d(a, b, c)}dt\,da\,db\,dc=-\iiiint\left[\lambda\left(\frac{dM_1}{da}+\frac{dM_2}{db}+\frac{dM_3}{dc}\right)\right.$$
$$\left.+\left(M_1\frac{d\lambda}{da}+M_2\frac{d\lambda}{db}+M_3\frac{d\lambda}{dc}\right)\right]\delta x\,dt\,da\,db\,dc.$$

The first term in brackets vanishes, and the second term is equal to $Jd\lambda/dx$,

whence $$-\iiiint\lambda\delta\frac{d(x, y, z)}{d(a, b, c)}dt\,da\,db\,dc$$
$$=\iiiint\left\{\frac{d\lambda}{dx}\delta x+\frac{d\lambda}{dy}\delta y+\frac{d\lambda}{dz}\delta z\right\}J\,dt\,da\,db\,dc.$$

Hence the conditions of the problem require that

$$\left.\begin{aligned}\ddot{x}+\frac{dV}{dx}-\frac{\rho_0}{\rho}\frac{d\lambda}{dx}&=0\\ \ddot{y}+\frac{dV}{dy}-\frac{\rho_0}{\rho}\frac{d\lambda}{dy}&=0\\ \ddot{z}+\frac{dV}{dz}-\frac{\rho_0}{\rho}\frac{d\lambda}{dz}&=0\end{aligned}\right\}\quad\text{..................(24).}$$

Now $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ are the component accelerations of the element whose coordinates are x, y, z, and are therefore equal to $\partial u/\partial t$, $\partial v/\partial t$, and $\partial w/\partial t$ respectively; and when we interpret $-\lambda\rho_0$ which must represent the pressure, equations (24) are the equations of motion in the ordinary form.

On Steady Motion.

37. When the motion is steady du/dt, dv/dt and dw/dt are each zero. In this case the general equations of motion can be integrated without having recourse to Clebsch's transformation. It will however be necessary to distinguish between irrotational and rotational motion.

The general equations of motion may be written,

$$\left.\begin{aligned}\frac{\partial u}{\partial t}=\frac{du}{dt}+\tfrac{1}{2}\frac{dq^2}{dx}-2v\zeta+2w\eta=\frac{dQ}{dx}\\ \frac{\partial u}{\partial t}=\frac{dv}{dt}+\tfrac{1}{2}\frac{dq^2}{dy}-2w\xi+2u\zeta=\frac{dQ}{dy}\\ \frac{\partial u}{\partial t}=\frac{dw}{dt}+\tfrac{1}{2}\frac{dq^2}{dz}-2u\eta+2v\xi=\frac{dQ}{dz}\end{aligned}\right\}\quad\ldots\ldots\ldots(25).$$

When the motion is steady and irrotational $\dot{u}$, $\dot{v}$, $\dot{w}$, ξ, η, ζ are each zero; whence, multiplying by dx, dy, dz, adding and integrating, we obtain

$$Q=\tfrac{1}{2}q^2-C,$$

or

$$\int\frac{dp}{\rho}+V+\tfrac{1}{2}q^2=C\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(26).$$

In this case the quantity C is evidently an absolute constant.

When the motion is rotational, let ds be an element of a stream line, then

$$u=q\frac{dx}{ds},\quad v=q\frac{dy}{ds},\quad w=q\frac{dz}{ds}.$$

Multiplying the general equations by u, v, w and adding,

we obtain

$$\frac{dQ}{ds}=\tfrac{1}{2}\frac{dq^2}{ds},$$

whence

$$\int\frac{dp}{\rho}+V+\tfrac{1}{2}q^2=A\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(27).$$

This is Bernoulli's Theorem.

Since we have integrated along a stream line, the quantity A is not an absolute constant, but a function of the parameter of a stream line: in other words if $\psi=$ const., $\chi=$ const. be two surfaces whose intersections determine the stream lines, A is a function of ψ and χ.

38. Let us now consider the steady motion of a liquid[1] which is symmetrical with respect to the axis of z. The vortex lines will evidently be perpendicular to every plane through the axis of z, hence by § 19 (41) the molecular rotation ω will be determined by the equation

$$2\omega=\frac{du}{dz}-\frac{dw}{d\varpi}.$$

[1] Stokes, "On the steady motion of incompressible fluids," *Trans. Camb. Phil. Soc.* vol. VII. p. 439.

Substituting for u and w their values in terms of Stokes' current function ψ, § 16 (25), we obtain

$$\frac{d^2\psi}{dz^2}+\frac{d^2\psi}{d\varpi^2}-\frac{1}{\varpi}\frac{d\psi}{d\varpi}+2\varpi\omega=0 \quad\text{............(28).}$$

The equations of motion are

$$\frac{dQ}{d\varpi}=u\frac{du}{d\varpi}+w\frac{du}{dz}=\tfrac{1}{2}\frac{d(q^2)}{d\varpi}+2w\omega,$$

$$\frac{dQ}{dz}=u\frac{dw}{d\varpi}+w\frac{dw}{dz}=\tfrac{1}{2}\frac{d(q^2)}{dz}-2u\omega.$$

Eliminating $Q-\frac{1}{2}q^2$, we obtain

$$u\frac{d\omega}{d\varpi}+w\frac{d\omega}{dz}+\omega\left(\frac{du}{d\varpi}+\frac{dw}{dz}\right)=0 \quad\text{...........(29).}$$

The equation of continuity § 9 (13) is

$$\frac{du}{d\varpi}+\frac{dw}{dz}+\frac{u}{\varpi}=0,$$

whence (29) becomes

$$u\frac{d\omega}{d\varpi}+w\frac{d\omega}{dz}-\frac{u\omega}{\varpi}=0,$$

or

$$\left(u\frac{d}{d\varpi}+w\frac{d}{dz}\right)\frac{\omega}{\varpi}=0 \quad\text{..................(30).}$$

Substituting the values of u, w and ω in terms of ψ, (30) becomes

$$\left(\frac{d\psi}{dz}\frac{d}{d\varpi}-\frac{d\psi}{d\varpi}\frac{d}{dz}\right)\left\{\frac{1}{\varpi^2}\left(\frac{d^2\psi}{dz^2}+\frac{d^2\psi}{d\varpi^2}-\frac{1}{\varpi}\frac{d\psi}{d\varpi}\right)\right\}=0 \quad\text{...(31).}$$

A first integral of this equation is evidently

$$\frac{d^2\psi}{dz^2}+\frac{d^2\psi}{d\varpi^2}-\frac{1}{\varpi}\frac{d\psi}{d\varpi}=\varpi^2 f(\psi) \quad\text{..............(32),}$$

whence by (28)

$$2\omega+\varpi f(\psi)=0 \quad\text{.......................(33).}$$

When the motion takes place in two dimensions, we shall, in exactly the same way, arrive at the equations

$$u\frac{d\zeta}{dx}+v\frac{d\zeta}{dy}=0,$$

and

$$\frac{d^2\psi}{dx^2}+\frac{d^2\psi}{dy^2}+2\zeta=0 \quad\text{....................(34),}$$

whence

$$\left(\frac{d\psi}{dy}\frac{d}{dx}-\frac{d\psi}{dx}\frac{d}{dy}\right)\left(\frac{d^2\psi}{dx^2}+\frac{d^2\psi}{dy^2}\right)=0 \quad\text{...........(35),}$$

a first integral of which is

$$\frac{d^2\psi}{dx^2} + \frac{d^2\psi}{dy^2} = f(\psi) \quad\text{.....................(36)},$$

whence by (34)

$$2\zeta + f(\psi) = 0 \quad\text{........................(37)}.$$

39. The subject of the steady motion of a liquid has been treated in the following manner by Clebsch[1].

Let b and c be any functions of x, y, z and t; then if the suffixes denote differentiation with respect to x, y and z, we may evidently put

$$u = b_y c_z - b_z c_y,\ \ v = b_z c_x - b_x c_z,\ \ w = b_x c_y - b_y c_x \quad\text{......(38)},$$

for these values of u, v and w satisfy the equation of continuity. From (38) we deduce

$$\left.\begin{array}{l} ub_x + vb_y + wb_z = 0 \\ uc_x + vc_y + wc_z = 0 \end{array}\right\} \quad\text{......................(39)},$$

hence the stream lines are the intersections of the surfaces $b = \text{const.}$, $c = \text{const.}$

Putting $$2T = u^2 + v^2 + w^2,$$

and multiplying equations (25) by dx, dy, dz respectively and adding, we obtain

$$dQ - dT = M_1 dx + M_2 dy + M_3 dz \quad\text{............(40)},$$

where $$M_1 = -v(v_x - u_y) + w(u_z - w_x) = -2v\zeta + 2w\eta,$$

with similar expressions for M_2 and M_3. From the values of M_1, M_2, M_3 it follows that

$$M_1 u + M_2 v + M_3 w = 0 \quad\text{..................(41)}.$$

Eliminating u, v, w from (39) and (41), we obtain

$$\begin{vmatrix} M_1, & b_x, & c_x \\ M_2, & b_y, & c_y \\ M_3, & b_z, & c_z \end{vmatrix} = 0.$$

Hence we may put

$$\left.\begin{array}{l} M_1 = Bb_x + Cc_x \\ M_2 = Bb_y + Cc_y \\ M_3 = Bb_z + Cc_z \end{array}\right\} \quad\text{.....................(42)},$$

where B and C are quantities whose values we shall hereafter determine; (40) may now be written

[1] *Crelle*, vol. LIV. p. 293.

$$dQ - dT = B\,(b_x dx + b_y dy + b_z dz) + C\,(c_x dx + c_y dy + c_z dz),$$

or
$$dQ - dT = Bdb + Cdc \quad\text{.....................(43).}$$

Since the left-hand side of (43) is a perfect differential, the right-hand side must be so also, whence if F be a function of b and c, we must have

$$B = \frac{dF}{db}, \quad C = \frac{dF}{dc} \quad\text{.....................(44),}$$

and therefore
$$Q - T = F\,(b, c) \quad\text{.....................(45)}$$

is an integral of the equations of motion.

When the motion is irrotational, M_1, M_2, M_3 and therefore B and C are each zero, and therefore F is an absolute constant.

40. We must now find the values of B and C. If we substitute the values of u, v and w from (38) in the expression for T and differentiate partially, we shall obtain

$$\frac{dT}{db_x} = -vc_z + wc_y,$$

$$\frac{dT}{db_y} = -wc_x + uc_z,$$

$$\frac{dT}{db_z} = -uc_y + vc_x,$$

whence
$$\frac{d}{dx}\left(\frac{dT}{db_x}\right) + \frac{d}{dy}\left(\frac{dT}{db_y}\right) + \frac{d}{dz}\left(\frac{dT}{db_z}\right)$$
$$= -c_x\,(w_y - v_z) - c_y\,(u_z - w_x) - c_z\,(v_x - u_y)$$
$$= -2\,(c_x\xi + c_y\eta + c_z\zeta).$$

From the first two of equations (42), we obtain

$$\begin{aligned} Bw &= M_1 c_y - M_2 c_x \\ &= 2c_y\,(-v\zeta + w\eta) - 2c_x\,(-w\xi + u\zeta) \\ &= 2w\,(c_x\xi + c_y\eta + c_z\zeta) \end{aligned}$$

by (39). Therefore

$$\frac{d}{dx}\left(\frac{dT}{db_x}\right) + \frac{d}{dy}\left(\frac{dT}{db_y}\right) + \frac{d}{dz}\left(\frac{dT}{db_z}\right) = -B = -\frac{dF}{db} \quad\text{....(46).}$$

Similarly

$$\frac{d}{dx}\left(\frac{dT}{dc_x}\right) + \frac{d}{dy}\left(\frac{dT}{dc_y}\right) + \frac{d}{dz}\left(\frac{dT}{dc_z}\right) = -C = -\frac{dF}{dc} \quad\text{....(47).}$$

41. By means of the preceding equations it can be shown that the conditions of steady motion make

$$\iiint (T-F)\,dx\,dy\,dz$$

a maximum or minimum.

For
$$\delta T = \frac{dT}{db_x}\,\delta b_x + \&c.,$$

and
$$\iiint \frac{dT}{db_x}\,\delta b_x\,dx\,dy\,dz = \iint \frac{dT}{db_x}\,\delta b\,dy\,dz - \iiint \frac{d}{dx}\left(\frac{dT}{db_x}\right)\delta b\,dx\,dy\,dz.$$

Whence, omitting the double integrals which refer to the boundary, we obtain

$$\begin{aligned}\iiint \delta T\,dx\,dy\,dz &= -\iiint \left\{\frac{d}{dx}\left(\frac{dT}{db_x}\right) + \frac{d}{dy}\left(\frac{dT}{db_y}\right) + \frac{d}{dz}\left(\frac{dT}{db_z}\right)\right\}\delta b\,dx\,dy\,dz \\ &\quad -\iiint \left\{\frac{d}{dx}\left(\frac{dT}{dc_x}\right) + \frac{d}{dy}\left(\frac{dT}{dc_y}\right) + \frac{d}{dz}\left(\frac{dT}{dc_z}\right)\right\}\delta c\,dx\,dy\,dz \\ &= \iiint \left\{\frac{dF}{db}\,\delta b + \frac{dF}{dc}\,\delta c\right\}dx\,dy\,dz\end{aligned}$$

by (46) and (47); whence

$$\iiint \delta(T-F)\,dx\,dy\,dz = 0,$$

which proves the proposition.

Impulsive Motion.

42. Let u, v, w and u', v', w' be the velocities of a fluid, just before, and just after the impulse; p the impulsive pressure. Then if S be any closed surface, the change of momentum parallel to x, of the fluid contained within S, must be equal to the component parallel to x of the impulsive pressure upon the surface of S.

Hence
$$\begin{aligned}\iiint \rho\,(u'-u)\,dx\,dy\,dz &= -\iint pl\,dS \\ &= -\iiint \frac{dp}{dx}\,dx\,dy\,dz.\end{aligned}$$

Therefore
$$\rho\,(u'-u) = -\frac{dp}{dx}$$

Similarly
$$\left.\begin{aligned}\rho\,(v'-v) &= -\frac{dp}{dy} \\ \rho\,(w'-w) &= -\frac{dp}{dz}\end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots(48).$$

Multiplying by dx, dy, dz and adding, we obtain

$$-\frac{dp}{\rho} = (u'-u)\,dx + (v'-v)\,dy + (w'-w)\,dz \ldots\ldots(49).$$

In the case of a liquid ρ is constant, whence differentiating (48) with respect to x, y, z respectively, and taking account of the equation of continuity, we obtain

$$\nabla^2 p = 0.$$

If the liquid were originally at rest it is clear that the motion produced by the impulse must be irrotational, whence if ϕ be its velocity potential, we must have

$$p = -\rho\phi \text{ ....(50).}$$

Examples and Applications.

43. *A mass of liquid whose external surface is a sphere of radius* a, *and which is subject to a constant pressure* Π, *surrounds a solid sphere of radius* b. *The solid sphere is annihilated, it is required to determine the motion of the liquid.*

It is evident that the only possible motion which can take place is one in which each element of liquid moves towards the centre, whence the free surfaces will remain spherical. Let R', R be their external and internal radii at any subsequent time, r the distance of any point of the liquid from the centre. The equation of continuity is

$$\frac{d}{dr}(r^2 \overset{v}{\cancel{V}}) = 0,$$

whence

$$r^2 v = F(t).$$

The equation for the pressure is

$$\frac{1}{\rho}\frac{dp}{dr} = -\frac{dv}{dt} - v\frac{dv}{dr}$$

$$= -\frac{F'(t)}{r^2} - \tfrac{1}{2}\frac{dv^2}{dr},$$

whence

$$\frac{p}{\rho} = A + \frac{F'(t)}{r} - \tfrac{1}{2}v^2,$$

when $r = R'$, $p = \Pi$, and when $r = R$, $p = 0$, whence if V, V' be the velocities of the internal and external surfaces

$$\frac{\Pi}{\rho} = F'(t)\left(\frac{1}{R'} - \frac{1}{R}\right) - \tfrac{1}{2}(V'^2 - V^2).$$

Since the volume of the liquid is constant,

$$R'^3 - R^3 = a^3 - b^3 = c^3,$$

also

$$F'(t) = \frac{d}{dt}(R^2 V),$$

whence

$$\frac{\Pi}{\rho} = V\frac{d}{dR}(R^2V)\left\{\frac{1}{(R^3+c^3)^{\frac{1}{3}}} - \frac{1}{R}\right\} - \tfrac{1}{2}V^2\left\{\frac{R^4}{(R^3+c^3)^{\frac{4}{3}}} - 1\right\}.$$

Putting $z = R^4V^2$, multiplying by $2R^2$ and integrating, we obtain

$$\tfrac{2}{3}\frac{\Pi(R^3-b^3)}{\rho R^4} = V^2\left\{\frac{1}{(R^3+c^3)^{\frac{1}{3}}} - \frac{1}{R}\right\},$$

which determines the velocity of the inner surface.

If the liquid had extended to infinity, we must put $c = \infty$, and we obtain

$$\frac{2\Pi}{3\rho}(b^3 - R^3) = R^3\left(\frac{dR}{dt}\right)^2,$$

whence if t be the time of filling up the cavity

$$t = \sqrt{\frac{3\rho}{2\Pi}}\int_0^b \frac{R^{\frac{3}{2}}\,dR}{\sqrt{b^3-R^3}}$$

$$= b\sqrt{\frac{\pi\rho}{6\Pi}}\,\frac{\Gamma(\frac{5}{6})}{\Gamma(\frac{4}{3})}.$$

The preceding example may be solved at once by the Principle of Energy.

The kinetic energy of the liquid is

$$2\pi\rho\int_R^{R'} r^2v^2dr = 2\pi\rho V^2R^4\int_R^{R'}\frac{dr}{r^2}$$

$$= 2\pi\rho V^2R^4\left\{\frac{1}{R} - \frac{1}{(R^3+c^3)^{\frac{1}{3}}}\right\}.$$

The work done by the external pressure is

$$4\pi\Pi\int_R^a r^2dr = \tfrac{4}{3}\Pi\pi\,(a^3 - R'^3)$$

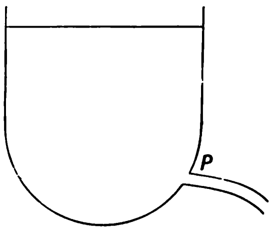
$$= \tfrac{4}{3}\Pi\pi\,(b^3 - R^3),$$

whence

$$\tfrac{2}{3}\Pi\,(b^3 - R^3) = V^2R^4\rho\left\{\frac{1}{R} - \frac{1}{(R^3+c^3)^{\frac{1}{3}}}\right\}.$$

44. The determination of the motion of a liquid in a vessel of any given shape is one of great difficulty, and the solution has been effected in only a comparatively few number of cases. If, however, liquid is allowed to flow out of a vessel, the inclinations of whose sides to the vertical are small, an approximate solution may be obtained by neglecting the horizontal velocity of the

liquid. This method of dealing with the problem is called the hypothesis of parallel sections.



Let us suppose that the vessel is kept full, and the liquid is allowed to escape by a small orifice at P. Let h be the distance of P below the free surface, and z that of any element of liquid. Since the motion is steady, the equation for the pressure will be

$$\frac{p}{\rho} - gz + \tfrac{1}{2}v^2 = C.$$

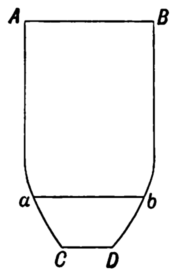
Now if the orifice be small in comparison with the area of the top of the vessel, the velocity at the free surface will be so small that it may be neglected; hence if Π be the atmospheric pressure, when $z = 0$, $p = \Pi$, $v = 0$ and therefore $C = \Pi/\rho$. At the orifice $p = \Pi$, $z = h$, whence the velocity of efflux is

$$v = \sqrt{2gh},$$

and is therefore the same as that acquired by a body falling from rest through a height equal to the depth of the orifice below the upper surface of the liquid. This result is called *Torricelli's Theorem.*

45. Let us in the next place suppose that the vessel is a surface of revolution, which has a finite horizontal aperture, and which is kept full[1].

Let A be the area of the top AB of the vessel, U the velocity of the liquid there; let K, u; Z, v be similar quantities for the aperture CD, and a section ab whose depth below AB is z: also let h be the depth of CD below AB.



The conditions of continuity require that

$$AU = Ku = Zv,$$

and since the horizontal motion is neglected, the equation for the pressure is

$$\frac{1}{\rho}\frac{dp}{dz} = g - \frac{dv}{dt} - v\frac{dv}{dz}.$$

Now U and u are functions of t alone, whilst Z is a function of z only, whence

$$\frac{dv}{dt} = \frac{A}{Z}\frac{dU}{dt} = \frac{K}{Z}\frac{du}{dt},$$

[1] Besant's *Hydromechanics.*

whence $$\frac{p}{\rho} = F(t) + gz - A\frac{dU}{dt}\int_0^z \frac{dz}{Z} - \tfrac{1}{2}v^2,$$

when $z = 0$, $p = \Pi$, $v = U$, therefore

$$\frac{\Pi}{\rho} = F(t) - \tfrac{1}{2}U^2,$$

when $z = h$, $p = \Pi$, $v = u$, whence if $a = \int_0^h Z^{-1}\,dz$,

$$\frac{\Pi}{\rho} = F(t) + gh - Aa\dot{U} - \tfrac{1}{2}u^2,$$

whence $$Aa\dot{U} = gh + \tfrac{1}{2}(U^2 - u^2)$$

$$= gh + \tfrac{1}{2}U^2\left(1 - \frac{A^2}{K^2}\right).$$

Putting $(A/K)^2 - 1 = 2m$, $2\sqrt{ghm} = \alpha a$, and integrating, we obtain

$$U = \sqrt{\frac{gh}{m}}\,\frac{C - \epsilon^{-\alpha t}}{C + \epsilon^{-\alpha t}},$$

where C is the constant of integration. Now initially $U = 0$ since the motion is supposed to begin from rest, therefore $C = 1$, and we obtain

$$U = \sqrt{\frac{gh}{m}}\tanh \tfrac{1}{2}\alpha t$$

$$= \sqrt{\frac{gh}{m}}\tanh t\sqrt{ghm}/a.$$

The velocity of efflux is

$$u = \sqrt{(1 + 2m)\frac{gh}{m}}\tanh t\sqrt{ghm}/a.$$

After a very long time has elapsed $\tanh t\sqrt{ghm}/a$ becomes equal to unity, and if K be very small compared with A, $m = \infty$, and we obtain Torricelli's Theorem

$$u = \sqrt{2gh}.$$

EXAMPLES.

1. A FINE tube whose section k is a function of its length s, in the form of a closed plane curve of area A, filled with ice is moved in any manner. When the component angular velocity of the tube about a normal to its plane is Ω, the ice melts without change of volume. Prove that the velocity of the liquid relatively to the tube at a point where the section is K at any subsequent time when ω is the angular velocity is

$$\frac{2Ac}{K}(\Omega - \omega),$$

where $1/c = \int k^{-1}ds$, the integral being taken once round the tube.

2. A centre of force attracting inversely as the square of the distance is at the centre of a spherical cavity within an infinite mass of liquid, the pressure on which at an infinite distance is ϖ, and is such that the work done by this pressure on a unit of area through a unit of length, is one half the work done by the attractive force on a unit of volume of the liquid from infinity to the initial boundary of the cavity; prove that the time of filling up the cavity will be

$$\pi a \sqrt{\frac{\rho}{\varpi}} \left\{2 - \left(\frac{3}{2}\right)^{\frac{3}{2}}\right\},$$

a being the initial radius of the cavity, and ρ the density of the liquid.

3. In the case of the steady motion of a gas issuing symmetrically and subject to no forces, neglecting changes of temperature; prove the following relation between the velocity v and the distance r from the centre;

$$4\pi v r^2 = Q \exp (v^2 - u^2)/2k,$$

where Q is the quantity of gas that issues per unit of time, k is the constant ratio of the pressure to the density, and u is the velocity at points where the pressure is k.

4. In the steady motion in one plane of a liquid under the action of natural forces, prove that

$$v\nabla^2 u - u\nabla^2 v = 0,$$

where $$\nabla^2 = d^2/dx^2 + d^2/dy^2.$$

5. Jets of water escape horizontally from orifices along a generating line of a vertical cylinder kept always full. Show that (to axes inclined at an angle $\frac{1}{4}\pi$ to the vertical) the equation of the lines of equal action for unit mass of water is of the form

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

Show also that the line of equal time for particles of water issuing simultaneously from the orifices, is the free path of the water which leaves the vessel by an orifice at a depth below the surface equal to that time.

6. A cistern discharges water into the atmosphere through a vertical pipe of uniform section. Show that air would be sucked in through a small hole in the upper part of the pipe, and explain how this result is consistent with an atmospheric pressure in the cistern.

7. A mass of liquid is moving so that the velocity at any point is proportional to the time, and the pressure is given by

$$p/\rho = \mu xyz - \tfrac{1}{2}t^2 (y^2z^2 + z^2x^2 + x^2y^2);$$

prove that this motion may have been generated from rest by finite natural forces independent of the time, with suitable boundary conditions: and show that if the direction of motion at every point coincides with the direction of the impressed force, each particle of liquid describes a curve which is the intersection of two hyperbolic cylinders.

8. Water is revolving with angular velocity ω in a smooth fine circular tube of radius a which it completely fills, and which rests on a horizontal plane. If the tube be made to revolve with uniform angular velocity ω' about a pivot O in its plane, show that the absolute angular velocity of the water round the centre C of the tube is unaltered. Also if ϖ be the average pressure of the water throughout the tube, show that the mean pressure of the water for a section through any point P of the tube is $\varpi + \mu ac\omega'^2 \cos\theta$, and that the resultant pressure on the tube at P per unit of length is $m\varpi/\mu a + ma\omega^2 + 2mc\omega'^2 \cos\theta$, where θ is the angle between OP and OC produced, $c = OC$, m is the mass of water which would fill a unit length of the tube, and μ that of a unit volume of water.

9. Prove that the equations of motion of a fluid referred to moving axes may be expressed in the form

$$\frac{1}{\rho}\frac{dp}{dx} - X + \frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz} - 2v\omega_3 + 2w\omega_2$$
$$- (\omega_2^2 + \omega_3^2)\, x - (\dot{\omega}_3 - \omega_1\omega_2)\, y + (\dot{\omega}_2 + \omega_3\omega_1)\, z = 0,$$

and two similar equations: where u, v, w are the component velocities of the fluid *relative* to the moving axes whose component angular velocities are ω_1, ω_2, ω_3.

10. A solid sphere of radius a is surrounded by a mass of liquid whose volume is $4\pi c^3/3$, and its centre is a centre of attractive force varying directly as the square of the distance. If the solid sphere be suddenly annihilated, show that the velocity of the inner surface when its radius is x, is given by

$$\dot{x}^2 x^3 \{(x^3 + c^3)^{\frac{1}{3}} - x\} = \left(\frac{2\Pi}{3\rho} + \frac{2}{9}\mu c^3\right)(a^3 - x^3)(c^3 + x^3)^{\frac{1}{3}},$$

where ρ is the density, Π the external pressure and μ the absolute force.

11. Prove that if ϖ be the impulsive pressure, ϕ, ϕ' the velocity potentials immediately before and after an impulse acts, V the potential of the impulses,

$$\varpi + \rho V + \rho(\phi' - \phi) = \text{const.}$$

12. A mass of compressible fluid is arranged in concentric spherical layers round a point under its own gravitation. Show that if radial vibrations be set up, the displacement z of an element is given by

$$\frac{1}{k\gamma\rho^{\gamma-1}}\frac{d^2z}{dt^2} = \frac{d^2z}{dr^2} + \left(\frac{\gamma}{\rho}\frac{d\rho}{dr} + \frac{2}{r}\right)\frac{dz}{dr} - 2\left(\frac{2-\gamma}{\rho}\frac{d\rho}{dr} + \frac{1}{r}\right)\frac{z}{r},$$

the pressure and density being connected by the equation $p = k\rho^\gamma$, and ρ in the differential equation being the density of the element when at rest.

13. If p denote the pressure at any point of a liquid moving irrotationally in two dimensions, under the action of a conservative system of forces, prove that

$$\nabla^2 \log \nabla^2 p = 0.$$

14. The surface of a vessel consists of two equal right cones, height $2c$, with coincident bases; it is fixed with its axis vertical and filled with water to half way up the axis of the upper cone, the

air above this level being initially at atmospheric pressure and the vessel closed. The water flows out of the vessel from a ring of apertures on the level of bisection of the axis of the lower cone. On the hypothesis of parallel sections, obtain a differential equation for the velocity of efflux, while the free surface is above the midway point, and show that one equation to find its maximum value in this stage is

$$u^2[1-\{c/(2c-x)\}^4]-2g\,(c+x)=2\overset{\Pi}{\cancel{\pi}}\,[\{c/(2c-x)\}^3-1]\,\rho^{-1},$$

where $x=$ height of surface above midway point.

15. If the motion of a homogeneous liquid be given by a single valued velocity potential, prove that the angular momentum of any spherical portion of the liquid about its centre is always zero.

16. Homogeneous liquid is moving so that

$$u=\gamma x+\alpha y,\ v=\beta x-\gamma y,\ w=0,$$

and a long cylindrical portion whose section is small and whose axis is parallel to the axis of z is solidified and the rest of the liquid destroyed. Prove that the initial angular velocity of the cylinder is

$$\frac{B\beta-A\alpha-2F\gamma}{A+B},$$

where A, B, F are the moments and products of inertia of the section of the cylinder about the axes.

17. Liquid is contained between two parallel planes; the free surface is an elliptic cylinder whose axis is perpendicular to the planes, and the semi-axes of whose section are a_1, b_1. All the liquid within a confocal elliptic cylinder, the semi-axes of whose section are a_2, b_2, is suddenly annihilated; prove that if Π be the pressure at the outer surface, the initial pressure at any point of the liquid is

$$\Pi\,\frac{\log(a+b)-\log(a_2+b_2)}{\log(a_1+b_1)-\log(a_2+b_2)},$$

where a and b are the semi-axes of a confocal cylinder through the point.

18. Fluid is contained within a sphere of small radius; prove that the momentum of the mass in the direction of the axis of x is greater than it would be if the whole were moving with the velocity at the centre by

$$\frac{Ma^2}{5\rho}\left\{\rho_x u_x+\rho_y u_y+\rho_z u_z+\tfrac{1}{2}\rho\nabla^2 u\right\}.$$

19. Obtain by means of Clebsch's transformation, § 39, the equations (33) and (37) of § 38.

20. Prove that when the motion of a liquid is steady, it is possible to draw a series of surfaces $P=$ const. each of which shall be covered with a network of stream lines and vortex lines. Prove also that at every point of such a surface

$$\frac{dP}{dn}=q\omega\sin\epsilon,$$

where q and ω are the resultant velocity and molecular rotation, and ϵ is the angle between their directions.

21. A cylindrical vessel of any form which is rotating about its axis, is filled with liquid which is rotating as a rigid body. Prove that if the cylinder be reduced to rest, the resulting motion of the liquid will be steady.

22. If the motion of a liquid be referred to axes moving with angular velocities θ_1, θ_2, θ_3 about themselves, prove that the components of molecular rotation are determined by the equation

$$\frac{d\xi}{dt}-\eta\theta_3+\zeta\theta_2+U\frac{d\xi}{dx}+V\frac{d\xi}{dy}+W\frac{d\xi}{dz}=\xi\frac{du}{dx}+\eta\frac{du}{dy}+\zeta\frac{du}{dz},$$

with two similar equations; where u, v, w are the component velocities of the liquid parallel to the moving axes, and U, V, W are its component velocities relative to these axes.

CHAPTER III.

ON SOURCES, DOUBLETS AND IMAGES.

46. WHEN the motion of a liquid is irrotational and symmetrical with respect to a fixed point, which we shall choose as the origin, the value of ϕ at any other point P is a function of the distance alone of P from the origin; and Laplace's equation becomes

$$\frac{d^2\phi}{dr} + \frac{2}{r}\frac{d\phi}{dr} = 0.$$

Therefore
$$\phi = -\frac{m}{r},$$

and
$$\frac{d\phi}{dr} = \frac{m}{r^2}.$$

The origin is therefore a singular point, from or to which the stream lines either diverge or converge, according as m is positive or negative. In the former case the singular point is called a *source*, in the latter case a *sink*.

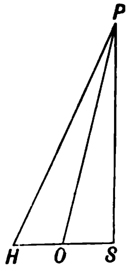
The flux across any closed surface surrounding the origin is

$$\iint \frac{d\phi}{dr}\, dS = \iint \frac{m \cos \epsilon}{r^2}\, dS = m \iint d\Omega$$
$$= 4\pi m,$$

where $d\Omega$ is the solid angle subtended by dS at the origin, and ϵ is the angle which the direction of motion makes with the normal to S drawn outwards.

The constant m is called the strength of the source.

47. A *doublet* is formed by the coalescence of an equal source and sink. To find its velocity potential; let there be a source and sink at S and H respectively, and let O be the middle point of SH, then



$$\phi = -\frac{m}{SP} + \frac{m}{HP}$$

$$= -\frac{mSH \cos SOP}{OP^2}.$$

Now let SH diminish and m increase indefinitely, but so that the product $m \, . \, SH$ remains finite and equal to μ, then

$$\phi = -\frac{\mu \cos SOP}{r^2}$$

$$= -\frac{\mu z}{r^3},$$

if the axis of z coincides with OS.

Hence the velocity potential due to a doublet is equal to the magnetic potential of a small magnet whose axis coincides with the axis of the doublet, and whose negative pole corresponds to the source end of the doublet.

48. The velocity potential due to a sheet of doublets of strength m per unit of surface, which is such that the axis of each doublet coincides with the direction of the normal to the sheet at the point at which it is situated, is

$$\phi = -\iint \frac{m \cos \epsilon}{r^2} dS$$

$$= -\iint m d\Omega.$$

If m be constant, $\phi = -m\Omega.$

Hence the velocity potential due to a doublet sheet is equal to the magnetic potential of a simple magnetic shell of strength $-m$.

49. When the motion is in two dimensions, and is symmetrical with respect to the axis of z, Laplace's equation becomes

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} = 0.$$

Therefore $\phi = m \log r,$

$$\frac{d\phi}{dr} = \frac{m}{r},$$

where r is the distance of any point from the axis. This value of ϕ represents a line source of infinite length, whose strength per unit of length is equal to m.

If ψ be the current function,

$$\frac{m}{r} = \frac{1}{r}\frac{d\psi}{d\theta}.$$

Therefore
$$\psi = m\theta$$
$$= m \tan^{-1}\frac{y}{x}.$$

The velocity potential due to a doublet in two dimensional motion is

$$\phi = m \log SP - m \log HP$$
$$= -m\frac{SH}{OP}\cos SOP = -\frac{\mu \cos SOP}{r}$$
$$= -\frac{\mu x}{r^2}.$$

Theory of Images.

50. Let H_1, H_2 be any two hydrodynamical systems situated in an infinite liquid. Since the lines of flow either form closed curves or have their extremities in the singular points or boundaries of the liquid, it will be possible to draw a surface S, which is not cut by any of the lines of flow, and over which there is therefore no flux, such that the two systems H_1, H_2 are completely shut off from one another.

The surface S may be either a closed surface such as an ellipsoid, or an infinite surface such as a paraboloid.

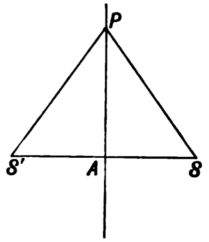
If therefore we remove one of the systems (say H_2) and substitute for it such a surface as S, everything will remain unaltered on the side of S on which H_1 is situated; hence the velocity of the liquid due to the combined effect of H_1 and H_2 will be the same as the velocity due to the system H_1 in a liquid which is bounded by the surface S.

The system H_2 is called the *image* of H_1 with respect to the surface S, and is such that if H_2 were introduced and S removed, there would be no flux across S.

The method of images was invented by Sir William Thomson,

and has been developed by Helmholtz, Maxwell and other writers[1]; it affords a powerful method of solving many important physical problems.

51. We shall now give some examples.



Let S, S' be two sources whose strengths are m. Through A the middle point of SS' draw a plane at right angles to SS'. The normal component of the velocity of the liquid at any point P on this plane is

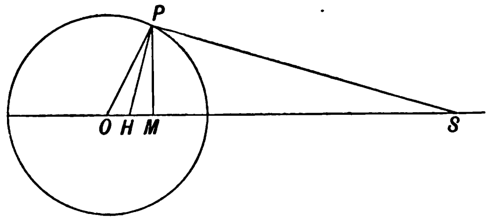
$$-\frac{m}{SP^2}\cos PSA + \frac{m}{S'P^2}\cos PS'A = 0.$$

Hence there is no flux across AP. If therefore Q be any point on the right-hand side of AP, the velocity potential due to a source at S, in a liquid which is bounded by the fixed plane AP, is

$$\phi = -\frac{m}{SQ} - \frac{m}{S'Q}.$$

Hence the image of a source S with respect to a plane is an equal source, situated at a point ~~H~~ S' on the other side of the plane, whose distance from it is equal to that of S.

52. *To find the image of a source placed outside a sphere*[2].



Let S be the source, O the centre of the sphere, a its radius, $OS = f$, $POS = \theta$, $\mu = \cos\theta$.

Then, if Φ be the velocity potential due to the source,

$$\Phi = -\frac{m}{SP} = -\frac{m}{(r^2 - 2fr\mu + f^2)^{\frac{1}{2}}}.$$

[1] Helmholtz, *Crelle*, vol. LV. 1858; Thomson, *Reprint of papers on Electricity and Magnetism*, p. 52; Maxwell, *Proc. Roy. Soc.*, 18 Feb. 1872; *Electricity and Magnetism*, vol. II. c. 12.

[2] W. M. Hicks, "On the Motion of Two Spheres in a Liquid," *Phil. Trans.* 1880.

Now at all points in the neighbourhood of the sphere, $r < f$; hence Φ can be expanded in a convergent series of the form

$$\Phi = -\frac{m}{f} - \frac{m}{f} \Sigma_1^\infty \left(\frac{r}{f}\right)^n P_n(\mu),$$

where $P_n(\mu)$ is the zonal harmonic of degree n.

At all points outside the sphere, the velocity potential of the image of S can be expanded in a series of the form

$$\Phi' = \frac{1}{r} \Sigma_0^\infty A_n \left(\frac{a}{r}\right)^n P_n.$$

Since the sphere is at rest, the surface condition is

$$\frac{d\Phi}{dr} + \frac{d\Phi'}{dr} = 0,$$

when $$r = a.$$

Therefore $$m\Sigma_1^\infty \frac{na^{n-1}}{f^{n+1}} P_n + \Sigma_0^\infty A_n \frac{(n+1)}{a^2} P_n = 0;$$

whence $A_0 = 0$,

$$A_n = -\frac{mn}{n+1}\left(\frac{a}{f}\right)^{n+1};$$

therefore

$$\Phi' = -m\Sigma_1^\infty \frac{n}{n+1} \frac{a^{2n+1}}{(fr)^{n+1}} P_n$$

$$= -\frac{ma}{f} \Sigma_1^\infty \frac{c^n}{r^{n+1}} P_n + \frac{ma}{f} \Sigma_1^\infty \frac{c^n}{r^{n+1}} \frac{P_n}{n+1} \;\ldots\ldots\ldots\ldots(1),$$

where $c = a^2/f$. Now if $c < r$,

$$\int_0^c \frac{d\lambda}{(r^2 - 2\lambda r\mu + \lambda^2)^{\frac{1}{2}}} = \Sigma_0^\infty \left(\frac{c}{r}\right)^{n+1} \frac{P_n}{n+1}.$$

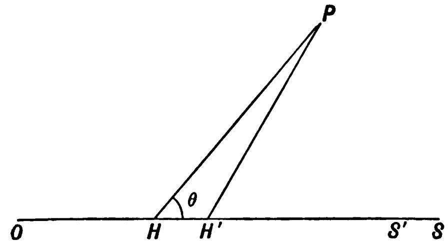
Hence, adding and subtracting ma/fr from (1), the value of Φ' may be written

$$\Phi' = -\frac{ma}{f} \cdot \frac{1}{(r^2 - 2rc\mu + c^2)^{\frac{1}{2}}} + \frac{m}{a}\int_0^c \frac{d\lambda}{(r^2 - 2\lambda r\mu + \lambda^2)^{\frac{1}{2}}}.$$

The first term represents a source of strength ma/f, situated at a point H such that $OH = c = a^2/f$, and which therefore coincides with the *electrical* image of S with respect to the sphere: the

second term represents a line sink of strength m/a per unit of length, extending from the inverse point H to the centre of the sphere.

53. *To find the image in a sphere of a doublet whose axis passes through the centre of the sphere.*



Let O be the centre of the sphere, a its radius, S a source of strength μ, S' an equal sink, and let H, H' be the inverse points of S, S'; also let $OS = f$, $HP = r$, $PHS = \theta$. Then, if ϕ be the velocity potential of the image,

$$\phi = -\frac{\mu a}{f} \cdot \frac{1}{HP} + \frac{\mu a}{f - SS'} \cdot \frac{1}{H'P} - \frac{\mu HH'}{aHP}.$$

But $OH \,.\, OS = OH' \,.\, OS' = a^2$, therefore

$$HH' = OH' - OH = a^2 \left(\frac{1}{OS'} - \frac{1}{OS}\right)$$

$$= \frac{a^2 SS'}{f^2},$$

also

$$H'P = HP - HH' \cos\theta.$$

Therefore

$$\phi = -\frac{\mu a}{fr} + \frac{\mu a}{fr}\left(1 + \frac{SS'}{f}\right)\left(1 + \frac{a^2 SS'}{f^2 r}\cos\theta\right) - \frac{\mu a SS'}{f^2 r}$$

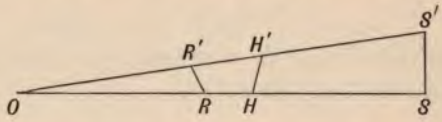
$$= \frac{\mu SS' a^3}{f^3 r^2}\cos\theta.$$

Now $\mu SS' = m$, where m is the strength of the original doublet, hence

$$\phi = m\left(\frac{a}{f}\right)^3 \frac{\cos\theta}{r^2}.$$

This is the velocity potential of a doublet situated at the inverse point H, whose strength $= -m\,(a/f)^3$.

54. *To find the image of a doublet whose axis is perpendicular to the line joining it with the centre of the sphere.*



Let S be a source, S' an equal sink; H, H' the inverse points S, S'. The image of S is a source of strength $\mu a/f$ at H, and a line sink of strength μ/a per unit of length from O to H.

Now $$HH' = \frac{SS'a^2}{f^2},$$

whence the source and sink at H, H' coalesce into a doublet at H of strength

$$\frac{\mu SS'a^3}{f^3} = \frac{ma^3}{f^3},$$

where m is the strength of the original doublet.

Let R, R' be any points on OH, OH', such that

$$OR \,.\, OS = OR' \,.\, OS';$$

then $$RR' = \frac{SS' \,.\, OR'}{f},$$

and the two sink and source elements at R, R' coalesce into a doublet of strength

$$-\frac{\mu}{a} \cdot \frac{SS' \,.\, OR}{f} = -\frac{m}{af} \,.\, OR.$$

Hence the image of S is a positive doublet at H of strength ma^3/f^3, together with a negative line doublet of strength $-mOR/af$ per unit of length, extending from O to H.

55. In the next place, let there be a source of strength m at a point P outside a sphere whose centre is O and radius a; and a line sink from P to Q, (where Q is a point on OP which lies on the side of P remote from O), of strength $-m/PQ$ per unit of length[1]. Let R be any point between P and Q; P', R', Q' the inverse points of P, R, Q. Also let $OR = x$, $OR' = y$.

[1] W. M. Hicks, "On the Problem of Two Pulsating Spheres in a Fluid," *Proc. Camb. Phil. Soc.* vol. III. p. 276.

The image of P consists of

(1) a source at P' of strength ma/OP,

(2) a line sink from O to P' of strength $-m/a$ per unit of length.

The image of the line sink element dx at R produces

(3) a sink at R', of strength

$$-\frac{madx}{PQ\,.\,x}=-\frac{mady}{PQ\,.\,y},$$

and

(4) a line source from O to R' of strength $mdx/PQ\,.\,a$ per unit of length.

In order to calculate the image of the line sink between P and Q, it will be convenient to consider first the portion of the image between O and Q', and secondly the portion between Q' and P'.

Since every element of PQ produces an elementary line source of strength $mdx/PQ\,.\,a$ between O and Q', the resultant is a line source between O and Q' whose strength per unit of length is

$$\int_{OP}^{OQ}\frac{mdx}{PQ\,.\,a}=\frac{m}{a}.$$

This line source cancels the portion of (2) which lies between O and Q'.

Only those elements of PQ which lie between P and R contribute anything to the ~~density~~ strength at R', hence, adding the effects of (2), (3) and (4), the total strength at R' is

$$\rho dy=-\frac{mdy}{a}-\frac{mady}{PQ\,.\,OR'}+\frac{mPRdy}{aPQ}=-\left(\frac{m}{a}+\frac{mOP}{a\,.\,PQ}\right)dy.$$

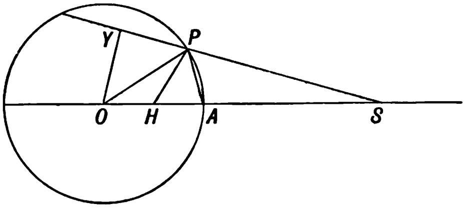
Therefore $$\rho=-\frac{m}{a}\left(1+\frac{OP}{PQ}\right)=-\frac{mOQ}{aPQ}.$$

But $$PQ=OQ-OP=\frac{a^2P'Q'}{OP'\,.\,OQ'}=\frac{P'Q'\,.\,OP\,.\,OQ}{a^2}.$$

Therefore $$\rho=-\frac{ma}{OP\,.\,P'Q'}.$$

Hence the image consists of a single source at P' of strength ma/OP, and a line sink from P' to Q' of strength $-ma/OP\,.\,P'Q'$ per unit of length; that is, the image is an arrangement of the same form as the object.

56. *To find the image of a source in a cylinder, the motion being in two dimensions.*



Let S be a source situated outside a cylinder, H the inverse point. Then, if an equal source be placed at H, the normal velocity q due to the combined effect of both is

$$q = -\frac{m}{SP}\cos OPY + \frac{m}{HP}\cos OPH.$$

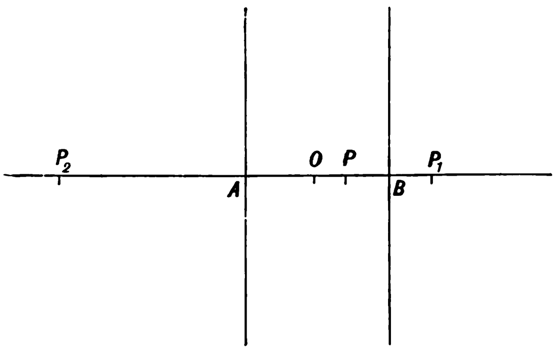
But since $OH . OS = OP^2$, the triangles OSP and OPH are similar, therefore

$$\begin{aligned} q &= -\frac{m}{SP}\cos OPY + \frac{m}{HP . OS}(SP + OP\cos OPY) \\ &= \frac{m}{OP}. \end{aligned}$$

Hence the image of a source at S is an equal source at the inverse point H, together with an equal sink at O the centre of the cylinder.

Similarly the image of a doublet is an equal doublet, but of opposite strength, situated at H.

57. We shall conclude this chapter by applying the method of images to determine the velocity potential due to a source situated between two infinite parallel planes[1].



[1] W. M. Hicks, *Quarterly Journal*, vol. xv. p. 274.

Let P be the source, and let the origin be the middle point O, of a line through P perpendicular to the two planes.

The image of P in the plane B will be another source P_1, such that $BP_1 = BP$; the image of P_1 in the plane A will be another source P_2 such that $AP_2 = AP_1$, and so on for an infinite series. Similarly the image of P in the plane A will be a source P'_1, such that $AP'_1 = AP$, and the image of P'_1 in B will be a source P'_2, such that $BP'_1 = BP'_2$, and so on. The velocity potential of the motion of the liquid contained between the two planes due to the source P, will be equal to the velocity potential of P together with the velocity potential of the two infinite trails of images.

Let $\qquad AB = 2a,\ OP = \xi,$

then
$$OP_1 = a + BP = 2a - \xi,$$
$$OP_2 = a + AP_1 = 4a - \xi,$$
$$OP_3 = a + BP_2 = 6a - \xi,$$
$$OP_n = 2na - \xi.$$

Similarly $\qquad OP'_n = 2na + \xi.$

(i) Let the motion be in three dimensions, and let z, ϖ be the coordinates of any point Q of the liquid; then

$$\phi = \frac{1}{\sqrt{\{(z-\xi)^2+\varpi^2\}}} + \Sigma_0^\infty \left[\frac{1}{\sqrt{\{(z+\xi-2a-4na)^2+\varpi^2\}}} + \frac{1}{\sqrt{\{(z+\xi+2a+4na)^2+\varpi^2\}}}\right]$$
$$+ \Sigma_1^\infty \left[\frac{1}{\sqrt{\{(z-\xi+4na)^2+\varpi^2\}}} + \frac{1}{\sqrt{\{(z-\xi-4na)^2+\varpi^2\}}}\right].$$

Therefore

$$\phi = \Sigma_{-\infty}^{\infty} \left[\frac{1}{\sqrt{\{(z+\xi-2a+4na)^2+\varpi^2\}}} + \frac{1}{\sqrt{\{(z-\xi+4na)^2+\varpi^2\}}}\right].$$

Each of these expressions is of the form $F(z, \varpi)$, where

$$F(z, \varpi) = \Sigma_{-\infty}^{\infty} \frac{1}{\sqrt{\{(z+4na)^2+\varpi^2\}}}.$$

We proceed to find a finite expression for this series. If a is positive,

$$\frac{1}{\sqrt{(a^2+b^2)}} = \frac{2a}{\pi}\int_0^{\frac{1}{2}\pi} \frac{d\theta}{a^2+b^2\cos^2\theta};$$

therefore

$$\frac{1}{\sqrt{\{(z+4na)^2+\varpi^2\}}}=\frac{2\varpi}{\pi}\int_0^{\frac{1}{2}\pi}\frac{d\theta}{(z+4na)^2\cos^2\theta+\varpi^2}$$

$$=\frac{1}{\pi\iota}\int_0^{\frac{1}{2}\pi}\left\{\frac{1}{(z+4na)\cos\theta-\varpi\iota}-\frac{1}{(z+4na)\cos\theta+\varpi\iota}\right\}d\theta.$$

Also $$\sin\frac{\pi\phi}{c}=\frac{\pi\phi}{c}\left(1-\frac{\phi^2}{c^2}\right)\dots\left(1-\frac{\phi^2}{n^2c^2}\right),$$

therefore, taking logarithms and differentiating, we obtain

$$\frac{\pi}{c}\cot\frac{\pi\phi}{c}=\frac{1}{\phi}+\Sigma_1^\infty\frac{2\phi}{\phi^2-n^2c^2}=\Sigma_{-\infty}^{\infty}\frac{1}{\phi+nc}.$$

Therefore $F(z, \varpi)$

$$=\frac{1}{\pi\iota}\int_0^{\frac{1}{2}\pi}\Sigma_{-\infty}^{\infty}\left\{\frac{1}{z\cos\theta-\varpi\iota+4na\cos\theta}-\frac{1}{z\cos\theta+\varpi\iota+4na\cos\theta}\right\}d\theta$$

$$=\frac{1}{4a\iota}\int_0^{\frac{1}{2}\pi}\left\{\cot\frac{\pi}{4a}(z-\varpi\iota\sec\theta)-\cot\frac{\pi}{4a}(z+\varpi\iota\sec\theta)\right\}\sec\theta\,d\theta$$

$$=\frac{1}{2a}\int_0^{\frac{1}{2}\pi}\frac{\sec\theta\sinh(\pi\varpi\sec\theta/2a)\,d\theta}{\cosh(\pi\varpi\sec\theta/2a)-\cos(\pi z/2a)}$$

$$=\frac{1}{\pi}\frac{d}{d\varpi}\int_0^{\frac{1}{2}\pi}\log\{\cosh(\pi\varpi\sec\theta/2a)-\cos(\pi z/2a)\}\,d\theta.$$

The first integral becomes infinite at the upper limit, but since the variable part of potential functions is the only part which it is necessary to consider, we may subtract $\frac{1}{2a}\int_0^{\frac{1}{2}\pi}\sec\theta\,d\theta$, which will make the integral finite, and we shall obtain

$$F(x, \varpi)=-\frac{1}{2a}\int_0^{\frac{1}{2}\pi}\frac{\exp(-\pi\varpi\sec\theta/2a)-\cos(\pi z/2a)}{\cosh(\pi\varpi\sec\theta/2a)-\cos(\pi z/2a)}\sec\theta\,d\theta.$$

And since $\phi=F\{(z+\xi-2a), \varpi\}+F\{z-\xi, \varpi\}$, we finally obtain

$$\phi=-\frac{1}{2a}\int_0^{\frac{1}{2}\pi}\left[\frac{\exp(-\pi\varpi\sec\theta/2a)+\cos\pi(z+\xi)/2a}{\cosh(\pi\varpi\sec\theta/2a)+\cos\pi(z+\xi)/2a}\right.$$
$$\left.+\frac{\exp(-\pi\varpi\sec\theta/2a)-\cos\pi(z-\xi)/2a}{\cosh(\pi\varpi\sec\theta/2a)-\cos\pi(z-\xi)/2a}\right]\sec\theta\,d\theta.$$

(ii) Let the motion be in two dimensions: writing x and y for z and ϖ respectively, we obtain

$$\phi=\tfrac{1}{2}\Sigma_{-\infty}^{\infty}\log\{(x+\xi-2a+4na)^2+y^2\}$$
$$+\tfrac{1}{2}\Sigma_{-\infty}^{\infty}\log\{(x-\xi+4na)^2+y^2\}.$$

Let $$f(x, y) = \log \Pi_{-\infty}^{\infty} \{(x + 4na)^2 + y^2\},$$
where the symbol Π denotes the infinite product for all positive and negative integral values of n including zero.

Now $$\sin\frac{\pi\theta}{c} = \frac{\pi\theta}{c}\left(1 - \frac{\theta^2}{c^2}\right) \dots\dots \left(1 - \frac{\theta^2}{n^2c^2}\right)$$
$$= \frac{\pi\theta}{c}\Pi'^{\infty}_{-\infty}\left(1 + \frac{\theta}{nc}\right),$$
where Π' denotes that the value $n = 0$ is excluded. Now
$$\Pi_{-\infty}^{\infty} \{(x + na)^2 - y^2\} = \Pi_{-\infty}^{\infty}(x + y + na)(x - y + na)$$
$$= (x + y)(x - y)\Pi'^{\infty}_{-\infty} n^2a^2\left(1 + \frac{x+y}{na}\right)\left(1 + \frac{x-y}{na}\right),$$
therefore
$$f(x, y) = \log \Pi'^{\infty}_{-\infty} 4^4n^2a^4 + \log\frac{(x + \iota y)}{4a}\Pi'^{\infty}_{-\infty}\left(1 + \frac{x + \iota y}{4na}\right)$$
$$+ \log\frac{x - \iota y}{4a}\Pi'^{\infty}_{-\infty}\left(1 + \frac{x - \iota y}{4na}\right).$$

The first term which is constant may be omitted; we therefore obtain
$$f(x, y) = \log \sin\frac{\pi}{4a}(x + \iota y)\sin\frac{\pi}{4a}(x - \iota y)$$
$$= \log(\cosh \pi y/2a - \cos \pi x/2a) - \log 2;$$
whence, omitting constant terms, the value of ϕ may be written
$$\phi = \tfrac{1}{2}\log\{\cosh \pi y/2a - \cos \pi(x - \xi)/2a\}$$
$$+ \tfrac{1}{2}\log\{\cosh \pi y/2a + \cos \pi(x + \xi)/2a\}.$$

EXAMPLES.

1. Prove that when the motion of a liquid is irrotational and symmetrical with respect to an axis, Stokes' current function satisfies the equation
$$\frac{d^2\psi}{dr^2} + \frac{\sin\theta}{r^2}\frac{d}{d\theta}\left(\operatorname{cosec}\theta\frac{d\psi}{d\theta}\right) = 0,$$
and thence show that the current function due to a source of strength m is
$$\psi = -m\cos\theta + \text{const.}$$

2. When the motion is in two dimensions, prove that the current function due to a source is $m\theta$, and apply this result to find the image of a source in a circular cylinder.

3. The motion of a liquid is in two dimensions, and there is a constant source at one point A in the liquid and an equal sink at another point B; find the form of the stream lines, and prove that the velocity at a point P varies as $(AP.BP)^{-1}$, the plane of the motion being unlimited.

If the liquid is bounded by the planes $x=0$, $x=a$, $y=0$, $y=a$, and if the source is at the point $(0, a)$ and the sink at $(a, 0)$, find an expression for the velocity potential.

4. The motion of a liquid in two dimensions is steady, and is due to the presence of any number of sources and sinks. If the mass of any source or sink be supposed equal to that of the liquid which it would generate per unit of time (the masses of the sinks being negative), show that any source has a tendency to move with an acceleration made up of accelerations from every other source and towards every sink, and proportional in each case to the numerical strength of the source and sink, and the inverse of its distance.

5. Liquid is bounded by two perpendicular planes OX, OY. A source is situated at a point whose distances from the planes are a and b respectively. Find the pressure at any point of either of the planes, (i) when the motion is in two dimensions, and (ii) when in three dimensions.

6. The boundary of a liquid consists of an infinite plane having a hemispherical boss, whose radius is a and centre O. A doublet of unit strength is situated at a point S, whose axis coincides with OS, where OS is perpendicular to the plane. P is any point on the plane, $OP=\varpi$, $OS=f$. Prove that the velocity of the liquid at P is

$$6f\varpi\left\{\frac{a^5}{(a^4+f^2\varpi^2)^{\frac{5}{2}}}-\frac{1}{(f^2+\varpi^2)^{\frac{5}{2}}}\right\}.$$

7. Prove that

$$\phi=f(t)\{(r^2+a^2-2az)^{-\frac{1}{2}}+(r^2+a^2+2az)^{-\frac{1}{2}}-r^{-1}\}+\psi(t)$$

is the velocity potential of a liquid, and interpret it. Find the surfaces of equal pressure if gravity in the negative direction of the axis of z be the only force acting.

8. Liquid enters a right circular cylindrical vessel by a supply pipe at the centre O and escapes by a pipe at a point A in the circumference; show that the velocity at any point P is proportional to $PB/PA \,.\, PO$, where B is the other end of the diameter AO. The vessel is supposed so shallow that the motion is in two dimensions.

9. A source is placed midway between two planes whose distance from one another is $2a$. Find the equation of the stream lines when the motion is in two dimensions; and show that those particles which at an infinite distance are distant $\frac{1}{2}a$ from one of the boundaries, issued from the source in a direction making an angle $\pi/4$ with it.

10. The boundaries of a liquid are given by $\theta = \pm\, \pi/2n$, and a source of strength m exists at the point $\theta = 0$, $r = a$. Prove that the current function for two dimensional motion is

$$\frac{m}{2\pi} \tan^{-1} \frac{r^{2n} \sin 2n\theta}{r^{2n} \cos 2n\theta - a^{2n}}.$$

11. A quantity of liquid moves in that quadrant of the plane of xy in which x and y are both positive, and which is bounded by the planes yz, zx: at the point $(a, 0)$ is a semicircular source of liquid, and at the origin a quadrantal sink. Assuming that the amount of liquid flowing out of the source per unit of time is equal to the amount which flows into the sink, and that the motion is in two dimensions; find the velocity potential, and prove that the general equation of the stream lines is

$$(x^2 + y^2)^2 - a^2 (x^2 - y^2) = \lambda a^2 xy.$$

CHAPTER IV.

VORTEX MOTION AND CYCLIC IRROTATIONAL MOTION.

58. THE most general kind of motion of which a fluid is capable is one which is a combination of rotational and irrotational motion; that is to say, the component velocities may be regarded as consisting of two parts, u_1, v_1, w_1 and u_2, v_2, w_2, where the former quantities are derivable from a velocity potential, whilst the latter, which depend upon the molecular rotation, cannot be so derived. The peculiarities of the motion specified by the latter quantities, and which depend upon the molecular rotation, were first investigated by Helmholtz[1] and will now be considered.

59. We have defined a vortex line to be a line whose direction coincides with the direction of the instantaneous axis of molecular rotation. If through every point of a small closed curve a series of vortex lines be drawn, they will enclose a mass of fluid which may be called a vortex filament, or shortly a vortex.

We have shown that if the forces which act on the fluid have a potential, and the density is a function of the pressure, the motion of the fluid constituting the vortex can never become irrotational. It will now be shown that every vortex possesses the following three fundamental properties:

(i) *Every vortex is always composed of the same elements of fluid.*

(ii) *The product of the angular velocity of any vortex into its cross section is constant with respect to the time, and is the same throughout its length.*

[1] *Crelle*, vol. LV. p. 25; translated by Tait, *Phil. Mag.* (4) XXXIII. p. 485.

(iii) *Every vortex must either form a closed curve or have its extremities in the boundaries of the fluid.*

Let a, b, c be the initial coordinates of an element of fluid whose coordinates at time t are x, y, z. Then

$$\frac{da}{\xi_0} = \frac{db}{\eta_0} = \frac{dc}{\zeta_0} = \frac{ds_0}{\omega_0} = \lambda.$$

But
$$\begin{aligned} dx &= \frac{dx}{da}da + \frac{dx}{db}db + \frac{dx}{dc}dc \\ &= \lambda\left(\xi_0\frac{dx}{da} + \eta_0\frac{dx}{db} + \zeta_0\frac{dx}{dc}\right) \\ &= \frac{\rho_0 \xi ds_0}{\rho\omega_0} \end{aligned}$$

by § 30 (10); hence
$$\frac{\rho_0 ds_0}{\omega_0} = \frac{\rho ds}{\omega} = \epsilon \qquad\qquad (1).$$

Let u, v, w be the component velocities at x, y, z; and let $u+du$, $v+dv$, $w+dw$ be the velocities at a neighbouring point $x+dx$, $y+dy$, $z+dz$ on the same vortex line. Since

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\zeta} = \frac{ds}{\omega} = \frac{\epsilon}{\rho},$$

therefore
$$\begin{aligned} du &= \frac{\epsilon}{\rho}\left(\xi\frac{du}{dx} + \eta\frac{du}{dy} + \zeta\frac{du}{dz}\right) \\ &= \frac{\epsilon}{\rho}\left(\xi\frac{du}{dx} + \eta\frac{dv}{dx} + \zeta\frac{dw}{dx}\right) \\ &= \epsilon\frac{\partial}{\partial t}\left(\frac{\xi}{\rho}\right) \end{aligned}$$

by § 24 (5).

The quantity du is the rate at which the projection of the element ds on the axis of x is increasing in length; and since this projection is equal to $\epsilon\partial(\rho^{-1}\xi)/\partial t$, the line ds still forms part of a vortex line.

This proves the first theorem.

To prove (ii) let σ be the area of the cross section at time t, then, since the mass of the element remains unchanged,

$$\rho_0\sigma_0 ds_0 = \rho\sigma ds.$$

Therefore by (1) $\qquad \sigma_0\omega_0 = \sigma\omega,$

which proves that $\sigma\omega$ is independent of the time.

Also, by § 7 and § 17 (26),

$$\iint (l\xi + m\eta + n\zeta)\, dS = \iiint \left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right) dx dy dz = 0,$$

$$\text{or } \iint \omega \cos \epsilon dS = 0,$$

where ϵ is the angle between the axis of rotation and the normal to S drawn outwards.

Now if we choose S so as to coincide with the surface of any finite portion of a vortex of small section, together with its two ends, $\cos \epsilon$ vanishes except at the two ends; and is equal to $+1$ at one end, and -1 at the other; hence

$$\omega_1 dS_1 - \omega_2 dS_2 = 0,$$

which proves the second part of (ii).

To prove the third theorem, we observe that if a vortex did not form a closed curve or have its extremities in the boundary, it would be possible to draw a closed surface cutting the vortex once only, and the surface integral would not vanish.

The first theorem and the first part of the second theorem depend on dynamical considerations; the second part of this theorem and the third theorem are kinematical.

60. Since every kind of motion may be regarded as a combination of rotational and irrotational motion, we may put

$$u = \frac{d\phi}{dx} + \frac{dN}{dy} - \frac{dM}{dz},$$

$$v = \frac{d\phi}{dy} + \frac{dL}{dz} - \frac{dN}{dx},$$

$$w = \frac{d\phi}{dz} + \frac{dM}{dx} - \frac{dL}{dy},$$

where ϕ is the velocity potential of that part of the motion which does not depend on the molecular rotation.

Hence in the case of a gas,

$$\nabla^2 \phi = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = -\frac{1}{\rho}\frac{\partial \rho}{\partial t},$$

but in the case of a liquid $\nabla^2 \phi = 0$: in addition to the above equations which ϕ must satisfy at every point of the fluid, ϕ must also be determined so as to satisfy the boundary conditions.

If we put $$J = \frac{dL}{dx} + \frac{dM}{dy} + \frac{dN}{dz},$$

we obtain $$2\xi = \frac{dw}{dy} - \frac{dv}{dz} = \frac{dJ}{dx} - \nabla^2 L,$$

with two similar equations. Hence if

$$J = 0 \text{ or a constant}$$

we have $$\nabla^2 L + 2\xi = 0,\ \nabla^2 M + 2\eta = 0,\ \nabla^2 N + 2\zeta = 0 \quad\text{.........(2)}.$$

It follows from (2) that if $J = 0$ or a constant, the quantities L, M, N are the potentials of distributions of matter whose densities are respectively equal to $\xi/2\pi$, $\eta/2\pi$, $\zeta/2\pi$; hence if x', y', z' be any point where molecular rotation exists, x, y, z any other point, and f the reciprocal of the distance between these two points, then

$$\left.\begin{aligned} L &= \frac{1}{2\pi}\iiint \xi' f dx' dy' dz' \\ M &= \frac{1}{2\pi}\iiint \eta' f dx' dy' dz' \\ N &= \frac{1}{2\pi}\iiint \zeta' f dx' dy' dz' \end{aligned}\right\} \quad\text{..................(3)},$$

where ξ', η', ζ' are the values of the components of molecular rotation at (x', y', z') and the integrations extend throughout those portions of fluid where there is vortex motion.

We have now to prove that the above values of L, M, N make $J = 0$ or a constant.

Since $$\frac{df}{dx} = -\frac{df}{dx'},$$

we have $$\begin{aligned} J &= -\frac{1}{2\pi}\iiint \left(\xi' \frac{df}{dx'} + \eta' \frac{df}{dy'} + \zeta' \frac{df}{dz'}\right) dx' dy' dz' \\ &= -\frac{1}{2\pi}\iint f\,(l\xi' + m\eta' + n\zeta')\, dS \\ &\quad + \frac{1}{2\pi}\iiint \left(\frac{d\xi'}{dx'} + \frac{d\eta'}{dy'} + \frac{d\zeta'}{dz'}\right) p\, dx' dy' dz'. \end{aligned}$$

The volume integral vanishes by § 17 (26), and if the vortices form closed rings the surface integral vanishes, since at the surface of each vortex $l\xi' + m\eta' + n\zeta' = 0$.

Also, if the fluid extends to infinity and is at rest there, the surface integral will either vanish or be equal to a constant, since ξ', η', ζ' and f all vanish at infinity. But if the fluid is bounded either externally or internally, and some of the vortices extend to this boundary and then break off, we must suppose the boundaries

removed and a hydrodynamical system substituted for them, such that the velocity at points occupied by the boundary remains unchanged. This hydrodynamical system will necessarily consist, in part, of the continuations of these vortices, which must either extend to infinity or form closed curves, and in either case the surface integral taken throughout the vortices included in this larger region, as well as throughout those included in the original region, will vanish or be constant.

61. If δu, δv, δw be the component velocities at a point x, y, z of a fluid due to an element $dx'dy'dz'$ whose rotations are ξ', η', ζ'; then

$$\delta u = \frac{1}{2\pi}\left(\zeta' \frac{df}{dy} - \eta' \frac{df}{dz}\right) dx' dy' dz',$$

whence if $r^{-1} = f$, we obtain

$$\left.\begin{aligned} \delta u &= \frac{1}{2\pi r^3}\{\eta'(z-z') - \zeta'(y-y')\}\, dx'dy'dz', \\ \delta v &= \frac{1}{2\pi r^3}\{\zeta'(x-x') - \xi'(z-z')\}\, dx'dy'dz', \\ \delta w &= \frac{1}{2\pi r^3}\{\xi'(y-y') - \eta'(x-x')\}\, dx'dy'dz', \end{aligned}\right\} \quad \ldots\ldots(4).$$

Hence, if q is the resultant velocity due to the element,

$$q = \frac{\omega \sin \epsilon}{2\pi r^2}\, dx'dy'dz' \ldots\ldots\ldots\ldots\ldots\ldots(5),$$

where ϵ is the angle which r makes with the direction of the axis of rotation of the vortex element. It also appears from (4), that this velocity is perpendicular to the plane containing the direction of r and the vortex element, and that its direction is that in which the point (x, y, z) would move if it were rigidly attached to a body moving with the vortex element.

62. At all points external to a vortex the motion is irrotational, and a velocity potential consequently exists. We shall now show that the velocity potential at any point, due to a vortex of small cross section, is proportional to the solid angle subtended by the vortex at that point.

Let x, y, z be the given point, x', y', z' any point on the vortex, r the distance between (x, y, z) and (x', y', z'). Using polar coordinates r, θ, χ referred to (x', y', z') as origin, we have

$$x - x' = r \sin\theta \cos\chi, \quad y - y' = r\sin\theta\sin\chi, \quad z - z' = r\cos\theta.$$

Now if Ω be the solid angle subtended at (x, y, z) by the vortex,

$$\begin{aligned}\Omega &= \iint \sin\theta\, d\theta\, d\chi \\ &= \int (1 - \cos\theta)\, d\chi \\ &= \int d\chi - \int \cos\theta \frac{d\chi}{ds}\, ds,\end{aligned}$$

where the integration with respect to s extends once round the vortex.

Now $$\frac{x - x'}{y - y'} = \cot\chi.$$

Therefore $$(y - y')\frac{dx'}{ds} - (x - x')\frac{dy'}{ds} = r^2 \sin^2\theta \frac{d\chi}{ds}.$$

Therefore

$$\Omega = \int d\chi - \int \frac{z - z'}{r}\left\{(y - y')\frac{dx'}{ds} - (x - x')\frac{dy'}{ds}\right\}\frac{ds}{(x - x')^2 + (y - y')^2}.$$

The first term is equal to 2π or zero according as the vortex does or does not embrace the axis of z; also

$$\frac{d\Omega}{dz} = -\int\left\{(y - y')\frac{dx'}{ds} - (x - x')\frac{dy'}{ds}\right\}\frac{ds}{r^3}.$$

Now by (4) if w be the z-component of the velocity due to a vortex of small cross section σ,

$$\frac{d\phi}{dz} = w = \frac{\omega\sigma}{2\pi}\int\left\{(y - y')\frac{dx'}{ds} - (x - x')\frac{dy'}{ds}\right\}\frac{ds}{r^3}.$$

Hence $$\frac{d\phi}{dz} = -\frac{\omega\sigma}{2\pi}\frac{d\Omega}{dz},$$

or $$\phi = -\frac{\omega\sigma}{2\pi}\Omega \quad \text{.........................(6).}$$

If the section of the vortex be of finite area, the velocity potential will be

$$\phi = -\frac{1}{2\pi}\iint \omega\Omega\, d\sigma \quad \text{....................(7),}$$

where the double integral extends over the cross section.

Since the solid angle Ω diminishes by 4π, whenever the point x, y, z describes a closed curve in the positive direction, which embraces the vortex once, ϕ is a many valued or cyclic function.

The product of the angular velocity and the cross section of a vortex filament, is called the *strength* of the filament.

Vortex Sheets.

63. If we have a sheet of thickness h, consisting of rotationally moving liquid, and if ω increase and h diminish indefinitely but so that the product ωh remains finite, equal to ω', we ultimately obtain an indefinitely thin film of rotationally moving fluid ~~liquid~~ whose molecular rotation is ω'. Such a film is called a *Vortex Sheet.*

By (3), if ξ', η', ζ' be the components of ω', the quantities L, M, N which determine the velocities are given by the equations

$$L = \frac{1}{2\pi}\iint \frac{\xi'}{R}\, dS, \quad M = \frac{1}{2\pi}\iint \frac{\eta'}{R}\, dS, \quad N = \frac{1}{2\pi}\iint \frac{\zeta'}{R}\, dS \ldots (8),$$

where R is the distance between any point on the vortex sheet and the point (x, y, z), and the integration extends over the vortex sheet.

64. It was first pointed out by Helmholtz[1], that the equations of motion and the equation of continuity of a perfect fluid do not exclude the possibility of slipping taking place along a surface; for the only conditions to which the velocity must be subject are, that it must be finite at all points of the fluid, other than points where sources or sinks exist, and also that its normal component at all points of any surface drawn in the fluid must be continuous. The above conditions obviously do not require that the tangential component should be the same on both sides of such a surface, and hence the conditions to which the velocity must be subject will not be violated if slipping takes place.

65. We shall now show that every surface of discontinuity over which slipping takes place has the properties of a vortex sheet.

Let l, m, n be the direction cosines of the normal at any point P of such a surface of discontinuity; u, v, w; u', v', w' the component velocities on the positive and negative sides of the surface. It is evident that it will be possible to draw a line in the tangent plane at P such that the tangential components along this line of the velocities on both sides of the surface shall be equal. Let λ', μ', ν' be the direction cosines of this line; and let λ, μ, ν be those

[1] *Phil. Mag.* Nov. 1868.

of the line through P perpendicular to l, m, n and λ', μ', ν', and which is therefore the line along which slipping must take place. Then

$$l(u-u')+m(v-v')+n(w-w')=0,$$
$$\lambda'(u-u')+\mu'(v-v')+\nu'(w-w')=0;$$

also let

$$\lambda(u-u')+\mu(v-v')+\nu(w-w')=\sigma.$$

From these equations we easily obtain

$$\frac{u-u'}{\lambda}=\frac{v-v'}{\mu}=\frac{w-w'}{\nu}=\sigma \quad\text{..................(9)}.$$

Let

$$L=\frac{1}{4\pi}\iint\frac{\sigma\lambda'}{R}\,dS \quad\text{.....................(10)},$$

the integration extending over the positive side of the sheet only; then

$$L=\frac{1}{4\pi}\iint(\nu m-\mu n)\frac{\sigma}{R}\,dS$$
$$=\frac{1}{4\pi}\iint\left\{m(w-w')-n(v-v')\right\}\frac{dS}{R}.$$

Now the surface S may be regarded as the limit of the surface of a solid bounded by S and another surface indefinitely near S whose distance from it is h; we may therefore write

$$L=\frac{1}{4\pi}\iint(mw-nv)\,dS-\frac{1}{4\pi}\iiint\left(w\frac{d}{dy}\frac{1}{R}-v\frac{d}{dz}\frac{1}{R}\right)dx\,dy\,dz$$

where the surface integral extends over the surface S and the surface indefinitely near it, and the volume integral extends throughout the volume enclosed by the two surfaces. The latter integral evidently vanishes in the limit. Integrating by parts we obtain

$$L=\frac{1}{4\pi}\iiint\frac{1}{R}\left(\frac{dw}{dy}-\frac{dv}{dz}\right)dx\,dy\,dz$$
$$=\frac{1}{2\pi}\iiint\frac{\xi}{R}\,dh\,dS$$
$$=\frac{1}{2\pi}\iint\frac{\xi'}{R}\,dS \quad\text{..............................(11)},$$

ultimately.

Comparing (10) and (11), we obtain

$$\xi'=\tfrac{1}{2}\sigma\lambda',\quad \eta'=\tfrac{1}{2}\sigma\mu',\quad \zeta'=\tfrac{1}{2}\sigma\nu',\quad \omega'=\tfrac{1}{2}\sigma.$$

It therefore follows that the effect of the surface of discontinuity is the same as that of a vortex sheet whose molecular rotation is $\frac{1}{2}\sigma$, and that the direction of the vortex lines is perpendicular to that of slipping.

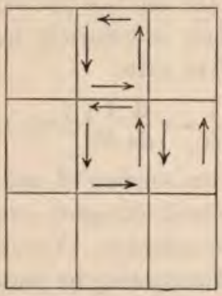
Circulation.

66. We have shown that the motion of a fluid may be separated into two kinds, rotational and irrotational motion; and it appears from § 62 that irrotational motion may be subdivided into two classes according as ϕ is a single valued or a many valued function. In the former case the motion is called *acyclic*, and in the latter case *cyclic* irrotational motion.

67. The line integral $\int(udx + vdy + wdz)$ taken along any curve joining a fixed point A, with a variable point P, is called the *flow* from A to P.

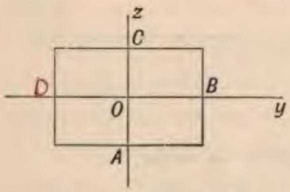
If the points A and P coincide, so that the curve along which the integration takes place is a closed curve, this line integral is called the *circulation* round the closed curve.

If any surface which is bounded by a closed curve be divided into elementary areas by a series of lines drawn upon it, the circulation round the bounding curve is equal to the sum of the circulations round each of the elementary areas; for the flow along the sides of all the elements, except those sides which form part of the boundary, is taken twice over and in opposite directions.



In the same way it can be shown that the circulation round any closed curve is equal to the sum of the circulations round its projections on the coordinate planes.

68. Let us now determine the circulation for an elementary rectangle $ABCD$, whose sides are dy, dz, the positive direction being from the axis of y to that of z.



Let x, y, z be the coordinates of O, the centre of inertia of the rectangle; u, v, w the velocities at O.

The portion of the circulation due to the two sides B and D is

$$(w + \tfrac{1}{2}w_y\,dy)\,dz - (w - \tfrac{1}{2}w_y\,dy)\,dz = w_y\,dy\,dz$$

and that due to the two sides C and $A = -v_z\,dy\,dz$.

Hence the circulation

$$= (w_y - v_z)\,dy\,dz.$$

Hence, if dS be an element of a surface S whose projection on the plane yz is the rectangle $ABCD$, the circulation round the boundary of S

$$= \iint [(w_y - v_z)\,dy\,dz + (u_z - w_x)\,dz\,dx + (v_x - u_y)\,dx\,dy].$$

Hence we obtain the following important analytical theorem, which is due to Prof. Stokes[1], viz.

$$\iint \left\{ l\left(\frac{dw}{dy} - \frac{dv}{dz}\right) + m\left(\frac{du}{dz} - \frac{dw}{dx}\right) + n\left(\frac{dv}{dx} - \frac{du}{dy}\right) \right\} dS$$
$$= \int (u\,dx + v\,dy + w\,dz) \quad \text{..................(12)},$$

where the surface integral is taken over any surface bounded by a given curve, and the line integral is taken once round the curve.

Substituting the quantities ξ, η, ζ, we obtain

$$2\iint (l\xi + m\eta + n\zeta)\,dS = \int (u\,dx + v\,dy + w\,dz) \quad \text{......(13)}.$$

69. Several important consequences can be deduced from this theorem.

If there are no vortices in the fluid, ξ, η, ζ are everywhere zero, and the circulation vanishes. Hence in this case ϕ must be a single valued function.

Since every vortex must either form a closed curve, or have its extremities in the boundaries of the fluid, it follows that if the circulation be taken round a closed curve which embraces a vortex once only, the surface S must cut the vortex an odd number of times. Hence in this case the circulation will not vanish, but will be equal to twice the surface integral on the left-hand side of (13). Since ξ, η, ζ are zero at all points of S, except those which lie in the vortex, the value of the circulation is $2\iint \omega \cos \epsilon\, d\sigma$ where $d\sigma$ is an element of that portion of S which is cut off by the vortex, ω the molecular rotation, and ϵ the angle which its direction makes with the normal to σ drawn outwards.

Hence the value of ϕ at any point P of a closed curve which embraces a vortex experiences a constant augmentation every time P travels round the curve to its original position, which is equal to twice the above-mentioned surface integral. This constant augmentation is called the cyclic constant of ϕ.

[1] Smith's Prize Examination, 1854.

If the line integral is taken round a closed curve which does not embrace a vortex, S can be drawn so as not to cut any of the vortices; if S cut any vortex once, it must cut it again, and by § 59 the two portions of the surface integral cancel one another; hence the surface integral and therefore the circulation round such a curve will be zero.

Since the circulation taken round any indefinitely thin vortex filament is equal to twice the product of its angular velocity and its cross section, it follows from § 59 that the circulation is independent of the time; and since every vortex of finite section can be divided into indefinitely thin vortex filaments, it follows that the circulation round a vortex of finite section is also independent of the time.

70. It thus appears that whenever there is circulation the velocity potential is such as would be due to some distribution of vortices in the fluid. These vortices need not however have an actual existence, since in the case of a liquid it is possible for hollow spaces to exist in the liquid round which circulation takes place; or the vortices of which ϕ is the velocity potential may lie beyond the boundaries of the fluid. For example, if $\phi = \tan^{-1} y/x = \theta$, ϕ is a two dimensional many valued velocity potential whose cyclic constant is 2π for all circuits which embrace the origin, and zero for all other circuits: and it will be shown in the second volume, that if the pressure at a distance from the origin be properly adjusted by means of suitable boundary conditions, it is possible for the cylinder $r = a$ to be a free surface, which forms the inner boundary of a *liquid*, and the space within which is devoid of liquid. It is also possible to have circulation round a fixed rigid cylinder, in which case ϕ will be the velocity potential of one of the possible motions of the liquid which may take place.

71. Since a fluid always flows from places of lower to places of higher velocity potential, it follows that when the motion is acyclic the stream lines cannot form closed curves but must begin or end in the boundaries or singular points of the fluid; but when the motion is cyclic some of the stream lines may be closed curves, whilst others begin and end in the boundaries of the fluid.

72. The circulation round any closed circuit may be shown not to alter with the time as follows[1].

[1] Sir W. Thomson, "Vortex Motion," *Trans. Roy. Soc. Edin.*, vol. xxv.

Let AB be a curve joining two points A and B of a fluid which always passes through the same elements of fluid; also let f be the tangential velocity of the fluid at any point P of AB; then

$$fds = udx + vdy + wdz;$$

therefore $$\frac{\partial}{\partial t}(fds) = \frac{\partial u}{\partial t}\,dx + u\,\frac{d\,(dx)}{dt} + \text{\&c.}$$

Let pq be the projection of ds on the axis of x; u, $u + du$ the component velocities of p and q parallel to x; then

$$u = dx/dt,\;\; u + du = d\,(x + dx)/dt;$$

hence $du = d\,.\,dx/dt$, therefore

$$\frac{\partial}{\partial t}\,(udx + vdy + wdz) = \frac{\partial u}{\partial t}\,dx + \frac{\partial v}{\partial t}\,dy + \frac{\partial w}{\partial z}\,dz + udu + vdv + wdw,$$

$$= d\,(Q + \tfrac{1}{2}q^2).$$

Therefore $$\frac{\partial}{\partial t}\int_A^B (udx + vdy + wdz) = [Q + \tfrac{1}{2}q^2]_B - [Q + \tfrac{1}{2}q^2]_A.$$

Since Q and q are always single valued functions, the right-hand side vanishes when the integration is taken round a closed curve, which proves the proposition.

73. If at some particular instant, which we shall choose as the origin of the time, the motion is irrotational and acyclic, the circulation will be zero round every closed circuit, and the preceding proposition shows that it will always remain zero.

Hence we obtain another proof of the proposition that motion which is once irrotational is always so; and also that irrotational motion which at any particular instant is acyclic, always remains so.

Simply and Multiply-Connected Regions.

74. Whenever the motion is cyclic, the flow between two points will not have the same value for two different lines joining them, unless the lines are such as are capable of being made to coincide, without cutting through any of the vortices or passing through the boundaries of the fluid. The latter class of lines are called *reconcileable* lines, the former *irreconcileable* lines.

75. We are thus led to consider the properties of simply and multiply-connected regions, which are defined as follows.

A *simply-connected* region, is one in which any two lines joining two given points, may be made to coincide with one another, without passing out of the region in question.

The spaces inside or outside an ellipsoid or paraboloid are simply connected regions.

A *multiply-connected* region, is one in which two or more lines can be drawn connecting two points, which cannot be made to coincide with each other without passing out of the region in question.

The space inside or outside an anchor ring, is an example of a doubly-connected region.

A region in which n irreconcileable lines can be drawn, is called an *n-ply* connected region.

Hence in a simply-connected region, every closed circuit is capable of being contracted to a point without passing out of the region. In an n-ply connected region, it is possible to draw $n-1$ different circuits, which cannot be contracted to a point or be made to coincide with one another without passing out of the region.

Any circuit drawn in a multiply-connected region, which is capable of being contracted to a point without passing out of the region, is called an *evanescible* circuit; and any two circuits which can be made to coincide with each other without passing out of the region, are called *mutually reconcileable.*

76. Every n-ply connected region, may be reduced to a simply connected region, by drawing $n-1$ barriers or diaphragms, such that each diaphragm meets every simple non-evanescible circuit once only. For example, the space outside two circles which do not cut one another, is a triply-connected region in two dimensions; but if from a point on each of the circles, we draw two lines to infinity which do not cut one another, the region becomes simply-connected.

77. If ϕ be a polycyclic velocity potential, the circulation round any closed curve, which does not cut any of the barriers is consequent[illegible] the circuit cuts all of the barriers once only, the circ[illegible] $+ \kappa_2 +$ &c. where κ_1, κ_2 are the cyclic constants [illegible] each barrier. The number of barriers which

must be drawn, in order to make the circulation round every closed curve vanish, is equal to the number of cyclic constants of ϕ.

78. Every polycyclic function may be expressed as the sum of the same number of monocyclic functions, as the function has cyclic constants. For if the number of cyclic constants be n there will be n simple non-evanescible circuits round which the circulation does not vanish; hence if

$$\phi = \kappa_1\Omega_1 + \kappa_2\Omega_2 + \ldots\ldots + \kappa_n\Omega_n,$$

where Ω_1, $\Omega_2 \ldots\ldots$ are monocyclic functions, whose cyclic constants are unity; and which are such that the line integral

$$\int\left(\frac{d\Omega_n}{dx}\frac{dx}{ds} + \frac{d\Omega_n}{dy}\frac{dy}{ds} + \frac{d\Omega_n}{dz}\frac{dz}{ds}\right)ds,$$

taken round any closed circuit is zero, except when the circuit cuts the barrier corresponding to κ_n, it follows that the circulations round each of the simple n non-evanescible circuits, are respectively equal to κ_1, $\kappa_2 \ldots\ldots$, hence the circulation round a circuit which cuts each barrier once only is equal to $\kappa_1 + \kappa_2 + \ldots\ldots + \kappa_n$.

Vorticity.

79. Let a mass of rotationally moving fluid be divided up into elementary vortex filaments; and let P be any point on the axis of one of these filaments, dm the mass of the filament which contains P, ω and dS the molecular rotation and cross section of the filament at P at time t. Then the quantity $\omega dS/dm$ is called the *vorticity* of the fluid at the point P.

This quantity has the same value at all points of the filament which contains P, and is constant with respect to the time, for if the suffixes denote the initial values of the quantities (or their values at some given epoch) and ds is an element of the axis of the vortex element, the vorticity

$$= \frac{\omega dS}{dm} = \frac{\omega_0 dS_0}{l_0\rho_0 dS_0} = \frac{\omega_0}{l_0\rho_0},$$

by § 59, (1); where l_0 is the initial length of the filament.

The *aggregate vorticity* of a mass M of rotationally moving fluid is equal to the sum of the vorticities of every filament, and therefore

$$= \frac{\Sigma\omega dS}{\Sigma\rho dS ds} = \frac{1}{M}\iint \omega \cos \epsilon dS,$$

where dS is an element of any surface which cuts all the vortex filaments once only, and ϵ is the angle between the direction of ω and the normal to S drawn outwards.

If the rotationally moving fluid is surrounded by irrotationally moving fluid, and consists of an arrangement such as a circular vortex ring, which is resolvable into elementary circular filaments which are perpendicular to the meridian sections of the ring, the aggregate vorticity is equal to $\frac{1}{2}\kappa/M$, where κ is the circulation round any closed circuit which embraces the ring once. But if the rotationally moving fluid consisted of the arrangement above described, together with an outer sheet which is resolvable into filaments lying in planes passing through the meridian sections of the ring, the circulation will remain unaltered, but the aggregate vorticity will be

$$\frac{\kappa}{2M_1} + \frac{1}{M_2}\iint \omega dS,$$

where M_1 is the mass of the inner ring, M_2 that of the sheet, and ω, dS are the molecular rotation and cross section at any point of one of the elementary filaments of the sheet. Hence the aggregate vorticity is not necessarily proportional to the circulation.

Green's Theorem.

80. The following theorem, which is of great importance in Electricity and various branches of physics, is due to Green[1].

Let ϕ and ψ be any two functions, which throughout the interior of a closed surface S are single valued, and which together with their first and second derivatives are finite and continuous at every point within S; then

$$\iiint \left(\frac{d\phi}{dx}\frac{d\psi}{dx} + \frac{d\phi}{dy}\frac{d\psi}{dy} + \frac{d\phi}{dz}\frac{d\psi}{dz}\right) dxdydz$$

$$= \iint \phi \frac{d\psi}{dn} dS - \iiint \phi\nabla^2\psi \, dxdydz \quad \ldots(14),$$

$$= \iint \psi \frac{d\phi}{dn} dS - \iiint \psi\nabla^2\phi \, dxdydz \quad \ldots.(15),$$

where the triple integrals extend throughout the volume of S, and the surface integrals over the surface of S, and dn denotes an element of the normal to S drawn outwards.

[1] *Mathematical Papers*, p. 24.

Integrating the left-hand side by parts, we obtain

$$\iiint \frac{d\phi}{dx}\frac{d\psi}{dx}\,dxdydz = \left[\iint \phi\,\frac{d\psi}{dx}\,dydz\right] - \iiint \phi\,\frac{d^2\psi}{dx^2}\,dxdydz. \quad (16),$$

where the brackets denote that the double integral is to be taken within proper limits. Now since the surface is a closed surface, any line parallel to x, which enters the surface a given number of times, must issue from it the same number of times; also the x-direction cosine of the normal at the point of entrance, will be of contrary sign to the same direction cosine at the corresponding point of exit; hence the surface integral

$$= \iint \phi\,\frac{d\psi}{dx}\,l dS.$$

Treating each of the other terms in a similar manner, we find that the left-hand side of (16)

$$= \iint \phi\,\frac{d\psi}{dn}\,dS - \iiint \phi\nabla^2\psi\,dxdydz.$$

The second equation (15) is obtained by interchanging ϕ and ψ.

81. We may deduce several important corollaries.

(i) Let ϕ be the velocity potential of a liquid, and let $\psi = 1$; then $\nabla^2\phi = 0$, and we obtain

$$0 = \iiint \nabla^2\phi\,dxdydz = \iint \frac{d\phi}{dn}\,dS \quad \text{...........(17)}.$$

The right-hand side is the analytical expression for the fact that the total flux across the closed surface is zero; in other words as much liquid enters the surface as issues from it.

(ii) Let ϕ and ψ be both velocity potentials, then

$$\iint \phi\,\frac{d\psi}{dn}\,dS = \iint \psi\,\frac{d\phi}{dn}\,dS \quad \text{...............(18)}.$$

(iii) Let $\phi = \psi$, where ϕ is the velocity potential of a liquid; then

$$\iiint \left\{\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 + \left(\frac{d\phi}{dz}\right)^2\right\} dxdydz = \iint \phi\,\frac{d\phi}{dn}\,dS \,..(19).$$

If we multiply both sides of (19) by $\frac{1}{2}\rho$, the left-hand side is equal to the kinetic energy of a liquid, and the equation shows that the kinetic energy of a liquid whose motion is acyclic and irrotational, which is contained within a closed surface, depends solely upon the motion of the surface.

82. Let us now suppose that liquid contained within such a surface is originally at rest, and let the liquid be set in motion by means of an impulsive pressure p applied to every point of the surface. The motion produced must be necessarily irrotational, and acyclic; also if ϕ be its velocity potential, it follows from § 42 (50) that $p = -\rho\phi$. Now the work done by an impulse, is equal to the product of the impulse into half the components in the direction of the impulse, of the initial and final velocities of the point to which it is applied; hence the work done,

$$= -\tfrac{1}{2}\iint p\,\frac{d\phi}{dn}\,dS = \tfrac{1}{2}\rho\iint \phi\,\frac{d\phi}{dn}\,dS,$$

and equation (19) asserts that the work done by the impulse is equal to the kinetic energy of the motion produced by it, which is a particular case of the Principle of Energy.

83. Let us in the next place suppose that liquid is contained within a closed surface which is in motion; and let the motion of the liquid be irrotational and acyclic; also let the surface be suddenly reduced to rest. Then if ϕ be the new velocity potential, $d\phi/dn = 0$, and therefore

$$\iiint \left\{\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 + \left(\frac{d\phi}{dz}\right)^2\right\} dx\,dy\,dz = 0,$$

whence $d\phi/dx$, $d\phi/dy$, and $d\phi/dz$ are each zero, and therefore the liquid is reduced to rest.

84. In proving Green's Theorem, we have supposed that the region through which we integrate, is contained within a single closed surface, but if the region were bounded externally and internally by two or more closed surfaces, the theorem would still be true, provided we take the surface integral with the positive sign over the external boundary, and with the negative sign over each of the internal boundaries.

85. Let us suppose that the liquid extends to infinity and is at rest there, and is bounded internally by one or more closed surfaces S_1, S_2 &c., and let us calculate the value of T for the space bounded by S_1, S_2 &c., and a very large sphere S whose centre is the origin. Then

$$T = \tfrac{1}{2}\rho\iint \phi\,\frac{d\phi}{dn}\,dS - \tfrac{1}{2}\rho\left[\iint \phi\,\frac{d\phi}{dn}\,dS\right],$$

where the square brackets indicate that the integral is to be taken over each of the internal boundaries.

Now at the surface of S, ϕ will be of the order m/r, where m is a constant, and $d\phi/dn = d\phi/dr = -m/r^2$; also if $d\Omega$ be the solid angle subtended by dS at the origin, $dS = r^2 d\Omega$; therefore

$$\iint \phi \frac{d\phi}{dn} dS = -\frac{m^2}{r}\iint d\Omega = -\frac{4\pi m^2}{r},$$

which vanishes when $r = \infty$. Hence the kinetic energy of an infinite liquid which is at rest at infinity, and which is bounded internally by closed surfaces is

$$T = -\tfrac{1}{2}\rho\left[\iint \phi \frac{d\phi}{dn} dS\right] \quad \ldots\ldots\ldots\ldots\ldots(20),$$

where the surface integral is to be taken over each of the internal boundaries.

The preceding expressions for the kinetic energy show that if the motion is acyclic and the *internal* boundaries of the liquid be suddenly reduced to rest, the whole liquid will be reduced to rest.

86. When the motion takes place in two dimensions, Green's Theorem may be established in a similar manner. Let the liquid be bounded externally by a closed surface S, and internally by one or more surfaces S_1, S_2.... Then

$$\iint\left(\frac{d\psi}{dx}\frac{d\phi}{dx} + \frac{d\psi}{dy}\frac{d\phi}{dy}\right) dxdy = \int \psi\left(\frac{d\phi}{dx}dy + \frac{d\phi}{dy}dx\right)$$
$$-\left[\int \psi\left(\frac{d\phi}{dx}dy + \frac{d\phi}{dy}dx\right)\right] - \iint \psi \nabla^2 \phi \, dxdy,$$

where $\nabla^2 = d^2/dx^2 + d^2/dy^2$ and the square brackets denote that the line integral is to be taken once round the circumferences of each of the internal boundaries. Now if we integrate round the boundary of the liquid in the contrary directions of the hands of a watch, the integration with respect to y will be in the same direction and that with respect to x in the opposite direction to s, whence the first integral becomes

$$\int \psi\left(\frac{d\phi}{dx}\frac{dx}{ds} - \frac{d\phi}{dy}\frac{dx}{ds}\right) ds,$$

also if dn be an element of the normal drawn outwards,

$$dx/ds = -dy/dn, \quad dy/ds = dx/dn,$$

whence $$\iint\left(\frac{d\phi}{dx}\frac{d\psi}{dx}+\frac{d\phi}{dy}\frac{d\psi}{dy}\right)$$

$$=\int\psi\frac{d\phi}{dn}\,ds-\left[\int\psi\frac{d\phi}{dn}\,ds\right]-\iint\psi\,\nabla^2\phi\,dxdy\;\ldots(21),$$

$$=\int\phi\frac{d\psi}{dn}\,ds-\left[\int\phi\frac{d\psi}{dn}\,ds\right]-\iint\phi\,\nabla^2\psi\,dxdy\;\ldots(22).$$

This is Green's Theorem for two-dimensional space.

Hence the kinetic energy of the liquid is

$$T=\tfrac{1}{2}\rho\int\phi\frac{d\phi}{dn}\,ds-\tfrac{1}{2}\rho\left[\int\phi\frac{d\phi}{dn}\,ds\right]\;\ldots\ldots\ldots(23).$$

In this equation ϕ may be either the velocity potential or the current function.

If the liquid extends to infinity and is at rest there, the value of ϕ if single valued, at a great distance from the origin, must be of the form

$$A\log r+r^{-1}(B\cos\theta+C\sin\theta),$$

and therefore when r is very large the first integral becomes equal to $2\pi\rho A^2\log r$ which becomes infinite when $r=\infty$ unless $A=0$; when this is the case, since all the other terms vanish, we obtain

$$T=-\tfrac{1}{2}\rho\left[\int\phi\frac{d\phi}{dn}\right]ds\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(24),$$

the integrations being taken round the internal boundaries only.

87. All the results of the last article may be also obtained by means of Stokes' theorem § 68 (12), and they may be extended to the case of polycyclic velocity potentials in the same way as in the next article. It should however be noticed that if ϕ be a polycyclic function, it will contain terms of the form $A\theta$, and hence ψ will contain terms of the form $A\log r$ and will therefore be single valued. We may therefore, in the case of cyclic motion, employ the single valued current function, instead of the velocity potential; but when there is circulation it follows from the last article that the kinetic energy will be infinite if the liquid extends to infinity. We shall show how the difficulty thus introduced may be evaded in Chapter VIII.

Thomson's Extension of Green's Theorem.

88. The proof of Green's Theorem given in § 80 holds good only when ϕ and ψ are single valued functions. If they are polycyclic functions, the surface and volume integrals on the right hand side of (14) and (15) become indeterminate. The extension of this theorem when ϕ and ψ are polycyclic functions is due to Sir W. Thomson[1].

Let us suppose that the region is multiply-connected, and that ϕ is a polycyclic function whose cyclic constants are $\kappa_1, \kappa_2\ldots$ Let the region be made simply connected by drawing the requisite number of barriers. Since we are not allowed to cross any barrier during the integration, we must include the surface on both sides of the barrier in the surface integrals. Hence if $d\sigma_1, d\sigma_2\ldots$ be elements of the different barriers corresponding to the quantities $\kappa_1, \kappa_2\ldots$ respectively

$$\iint \phi \frac{d\psi}{dn}\, dS = \iint \phi \frac{d\psi}{dn}\, dS + \iint \phi \frac{d\psi}{dn}\, d\sigma_1 + \ldots\ldots$$

where on the right-hand side, the integration with respect to S extends over the boundaries, and that with respect to σ_1 over *both* sides of the barrier σ_1.

Now the values of $d\psi/dn$ are equal in magnitude and of contrary sign at two contiguous points situated on different sides of the barrier, also the value of ϕ on the negative side exceeds that on the positive side by κ_1, therefore

$$\iint \phi \frac{d\psi}{dn}\, d\sigma_1 = \iint \phi \frac{d\psi}{dn}\, d\sigma_1 - \iint (\phi + \kappa_1) \frac{d\psi}{dn}\, d\sigma$$

$$= -\kappa_1 \iint \frac{d\psi}{dn}\, d\sigma_1,$$

where the integration on the left-hand side extends over both sides of the barriers, and that on the right over the positive side only.

Hence instead of $\iint \phi\, d\psi/dn\,.\,dS$, we must write

$$\iint \phi \frac{d\psi}{dn}\, dS - \kappa_1 \iint \frac{d\psi}{dn}\, d\sigma_1 - \kappa_2 \iint \frac{d\psi}{dn}\, d\sigma_2 - \ldots\ldots$$

[1] "On Vortex Motion," *Trans. Roy. Soc. Edin.*, vol. xxv. p. 217.

Similarly if ψ were a polycyclic function whose cyclic constants are κ'_1, κ'_2...... we must write instead of $\iint \psi d\phi/dn \, . \, dS$,

$$\iint \psi \frac{d\phi}{dn} dS - \kappa'_1 \iint \frac{d\phi}{dn} d\sigma_1 - \kappa'_2 \iint \frac{d\phi}{dn} d\sigma_2 - \ldots\ldots$$

Hence when ϕ and ψ are polycyclic functions and the region is a multiply-connected one, Green's Theorem becomes

$$\iiint \left(\frac{d\phi}{dx}\frac{d\psi}{dx} + \frac{d\phi}{dy}\frac{d\psi}{dy} + \frac{d\phi}{dz}\frac{d\psi}{dz} \right) dx\,dy\,dz$$

$$= \iint \phi \frac{d\psi}{dn} dS - \left[\iint \phi \frac{d\psi}{dn} dS \right] + \kappa_1 \iint \frac{d\psi}{dn} d\sigma_1 + \kappa_2 \iint \frac{d\psi}{dn} d\sigma_2 + \ldots\ldots$$
$$- \iiint \phi \nabla^2 \psi \, dx\,dy\,dz \ldots\ldots\ldots\ldots (25),$$

$$= \iint \psi \frac{d\phi}{dn} dS - \left[\iint \psi \frac{d\phi}{dn} dS \right] + \kappa'_1 \iint \frac{d\phi}{dn} d\sigma_1 + \kappa'_2 \iint \frac{d\phi}{dn} d\sigma_2 + \ldots\ldots$$
$$- \iiint \psi \nabla^2 \phi \, dx\,dy\,dz \ldots\ldots\ldots\ldots (26),$$

where the first integrals on the right hand side are to be taken over the outer boundary, and the square brackets denote that the second integrals are to be taken over each of the internal boundaries.

89. Putting $\phi = \psi$, it follows that if the liquid extends to infinity and is at rest there,

$$T = -\tfrac{1}{2}\rho \left[\iint \phi \frac{d\phi}{dn} dS \right] + \tfrac{1}{2}\kappa_1\rho \iint \frac{d\phi}{dn} d\sigma_1 + \tfrac{1}{2}\kappa_2\rho \iint \frac{d\phi}{dn} d\sigma_2 + \ldots (27).$$

The first term represents the work done by the impulsive pressure which must be applied to the boundaries S in order to produce the actual motion from rest. The second term represents the work done by a uniform impulsive pressure $\kappa_1\rho$, applied in the positive direction to every point of the barrier corresponding to κ_1. Hence cyclic irrotational motion may be artificially generated by means of a proper impulsive pressure applied to every point of the boundaries, together with uniform impulsive pressures $\kappa_1\rho$, $\kappa_2\rho$......, applied respectively to every point of the barriers, which must be drawn in order to make the region occupied by the liquid simply connected. We may therefore generalise the theorem of § 85, by asserting *that if irrotationally moving liquid occupying a multiply-connected space, is bounded by moving surfaces, which are suddenly brought to rest, the whole liquid will be reduced*

to rest unless its motion is cyclic; and that in the latter case, the cyclic motion which could have been generated in the manner above described will not be destroyed.

90. The foregoing arguments show *that if the bounding surface of a liquid which was originally at rest, be made to vary in a given arbitrary manner, the kinetic energy of the liquid at each instant, will be less than it would be if the liquid had any other motion consistent with the given motion of the bounding surface.*

Since the liquid is originally at rest, the motion which is caused by the variation of the bounding surface will be acyclic irrotational motion. But the most general kind of motion which is possible within the surface is a combination of acyclic, cyclic irrotational motion, and vortex motion. The first can be destroyed by means of a suitable impulsive pressure applied to every point of the boundary, but the two latter cannot be destroyed by any operations performed on the boundary alone. Hence the kinetic energy of the acyclic motion alone, must always be less than the kinetic energy of the most general possible motion.

This theorem is due to Sir W. Thomson[1].

91. When the motion is rotational the kinetic energy cannot be obtained by Green's Theorem, since within a vortex there is no velocity potential. In this case

$$\begin{aligned} T &= \tfrac{1}{2}\rho \textstyle\iiint (u^2+v^2+w^2)\,dx\,dy\,dz, \\ &= \tfrac{1}{2}\rho \iiint\left\{u\left(\frac{d\phi}{dx}+\frac{dN}{dy}-\frac{dM}{dz}\right)+v\left(\frac{d\phi}{dy}+\frac{dL}{dz}-\frac{dN}{dx}\right)\right. \\ &\qquad\qquad \left.+\,w\left(\frac{d\phi}{dz}+\frac{dM}{dx}-\frac{dL}{dy}\right)\right\}dx\,dy\,dz, \end{aligned}$$

by § 60. Integrating by parts, the terms involving ϕ

$$= \tfrac{1}{2}\rho \textstyle\iint \phi\,(lu+mv+nw)\,dS,$$

since the volume integral vanishes by virtue of the equation of continuity. The other terms

$$\begin{aligned} &= \tfrac{1}{2}\rho \textstyle\iint \{L(nv-mw)+M(lw-nu)+N(mu-lv)\}\,dS, \\ &+ \tfrac{1}{2}\rho \iiint\left\{L\left(\frac{dw}{dy}-\frac{dv}{dz}\right)+M\left(\frac{du}{dz}-\frac{dw}{dx}\right)+N\left(\frac{dv}{dx}-\frac{du}{dy}\right)\right\}dx\,dy\,dz. \end{aligned}$$

[1] "Notes on Hydrodynamics," *Camb. and Dubl. Math. Journ.*, vol. IV. p. 90.

If the liquid extends to infinity and is at rest there, and all the vortices are within a finite distance of the origin, the surface integrals will vanish and we obtain

$$T = \rho \iiint (L\xi + M\eta + N\zeta)\, dx\, dy\, dz \quad\text{............(28).}$$

92. Let us now suppose that we have two closed vortices of small cross sections σ_1, σ_2. Let ds_1, ds_2 be elements of their lengths; κ_1, κ_2 the circulations due to them; then

$$T = \tfrac{1}{2}\kappa_1\rho \int\left(L\frac{dx}{ds_1} + M\frac{dy}{ds_1} + N\frac{dz}{ds_1}\right) ds_1$$

$$+ \tfrac{1}{2}\kappa_2\rho \int\left(L\frac{dx}{ds_2} + M\frac{dy}{ds_2} + N\frac{dz}{ds_2}\right) ds_2,$$

where the line integrals extend round each respective vortex. Now

$$L = \frac{\kappa_1}{4\pi}\int \frac{1}{r}\frac{dx}{ds_1'}\, ds_1' + \frac{\kappa_2}{4\pi}\int \frac{1}{r}\frac{dx}{ds_2'}\, ds_2'; \text{ \&c. \&c.}$$

Therefore $$T = \frac{\rho}{8\pi}\left(A\kappa_1^2 + 2B\kappa_1\kappa_2 + C\kappa_2^2\right)$$

where $$A = \iint \frac{1}{r}\left(\frac{dx}{ds_1}\frac{dx}{ds_1'} + \frac{dy}{ds_1}\frac{dy}{ds_1'} + \frac{dz}{ds_1}\frac{dz}{ds_1'}\right) ds_1\, ds_1',$$

$$B = \iint \frac{1}{r}\left(\frac{dx}{ds_1}\frac{dx}{ds_2} + \frac{dy}{ds_1}\frac{dy}{ds_2} + \frac{dz}{ds_1}\frac{dz}{ds_2}\right) ds_1\, ds_2,$$

and C is obtained from A by changing s_1, s_1' into s_2, s_2'. If ϵ be the angle between the two elements ds_1, ds_2, these expressions may be written

$$A = \iint \frac{\cos\epsilon}{r}\, ds_1 ds_1', \quad B = \iint \frac{\cos\epsilon}{r}\, ds_1 ds_2, \quad C = \iint \frac{\cos\epsilon}{r}\, ds_2 ds_2'.$$

The quantities A and C are evidently the coefficients of self-induction of two electric currents of unit strengths which coincide with the vortices κ_1 and κ_2 respectively, and the quantity B is the coefficient of mutual induction of two such currents. Hence the kinetic energy of the hydrodynamical system is equal to the electro-kinetic energy of two currents of strengths $\frac{1}{2}\kappa_1(\rho/\pi)^{\frac{1}{2}}$ and $\frac{1}{2}\kappa_2(\rho/\pi)^{\frac{1}{2}}$ respectively, which occupy the positions of the vortices. This proposition may easily be extended to any number of vortices.

93. Another expression for T may be obtained in the form

$$T = 2\rho \iiint \{u\,(y\zeta - z\eta) + v\,(z\xi - x\zeta) + w\,(x\eta - y\xi)\}\,dx\,dy\,dz \ldots (29).$$

For the first term

$$= \rho \iiint u \left\{ y\left(\frac{dv}{dx} - \frac{du}{dy}\right) - z\left(\frac{du}{dz} - \frac{dw}{dx}\right)\right\} dx\,dy\,dz$$

$$= -\rho \iiint \left\{(vy + wz)\frac{du}{dy} - u^2\right\} dx\,dy\,dz,$$

since the surface integral vanishes. Transforming the other terms in the same way, adding, and making use of the equation of continuity, we obtain

$$\rho \iiint \left(u^2 + v^2 + w^2 + xu\frac{du}{dx} + yv\frac{dv}{dy} + zw\frac{dw}{dz}\right) dx\,dy\,dz.$$

Integrating the last three terms by parts, the right hand side of (29)

$$= \tfrac{1}{2}\rho \iiint (u^2 + v^2 + w^2)\,dx\,dy\,dz.$$

94. When the motion is symmetrical with respect to the axis of z, an expression for T may be obtained in terms of Stokes' current function; for

$$T = \pi\rho \iint \frac{1}{\varpi}\left\{\left(\frac{d\psi}{d\varpi}\right)^2 + \left(\frac{d\psi}{dz}\right)^2\right\} d\varpi\,dz.$$

Therefore

$$\frac{T}{\pi\rho} = \int \frac{\psi}{\varpi}\left(\frac{d\psi}{d\varpi}dz + \frac{d\psi}{dz}d\varpi\right) - \left[\int \frac{\psi}{\varpi}\left(\frac{d\psi}{d\varpi}dz + \frac{d\psi}{dz}d\varpi\right)\right]$$
$$- \iint \frac{\psi}{\varpi}\left(\frac{d^2\psi}{d\varpi^2} - \frac{1}{\varpi}\frac{d\psi}{d\varpi} + \frac{d^2\psi}{dz^2}\right) d\varpi\,dz,$$

where the first integral refers to the external, and the second integral to the internal boundaries of the liquid.

Now in order that this kind of motion may be possible, it is necessary that the boundaries should be surfaces of revolution whose axes coincide with the axis of z. Let s be an element of the meridian curve of one of the boundaries, and let the integration with respect to s be taken from z to ϖ. Since the integration with respect to ϖ will be in the same direction, and that with respect to z in the opposite direction to s, the first integral becomes

$$\int \frac{\psi}{\varpi}\left(\frac{d\psi}{dz}\frac{d\varpi}{ds} - \frac{d\psi}{d\varpi}\frac{dz}{ds}\right) ds = \int \frac{\psi}{\varpi}\frac{d\psi}{dn}\,ds,$$

where dn is an element of the normal drawn outwards. The volume integral is equal to

$$-2\iint \psi \omega \, d\varpi \, dz,$$

where ω is the molecular rotation: whence

$$T = \pi\rho \int \frac{\psi}{\varpi} \frac{d\psi}{dn} \, ds - \pi\rho \left[\int \frac{\psi}{\varpi} \frac{d\psi}{dn} \, ds\right] + 2\iint \psi \omega \, d\varpi \, dz \; \ldots\ldots (30).$$

If the motion is irrotational and the liquid extends to infinity, and is at rest there,

$$T = -\pi\rho \left[\int \frac{\psi}{\varpi} \frac{d\psi}{dn} \, ds\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots (31),$$

where the integration is taken once round the meridian curves of each of the internal boundaries.

On the Connection between Vortex Motion and Electromagnetism.

95. In § 60, we have shown that the velocity potential at P due to a single closed vortex filament of strength m, is

$$\phi = -m\Omega/2\pi,$$

where Ω is the solid angle subtended by the vortex at P.

This is the magnetic potential of an electric current of strength $-m/2\pi$, which flows round a closed circuit coinciding with the vortex (Maxwell, *Electricity and Magnetism*, vol. II. §§ 410 and 484). Now the magnetic potential due to such a current is the same as that due to a simple magnetic shell of strength $-m/2\pi$ which is bounded by the current; also by § 48, ϕ is the velocity potential due to a doublet sheet of strength $m/2\pi$ bounded by the vortex. Hence a vortex filament and a doublet sheet respectively correspond to an electric current and a magnetic shell, and a vortex sheet may be replaced by a doublet sheet in the same manner as an electric current may be replaced by a magnetic shell.

The action of a vortex filament upon the surrounding liquid is determined by the quantities L, M, N, whence it follows from (3) that the molecular relation corresponds to an electric current: the quantities L, M, N to the components F, G, H of electromagnetic momentum; and the velocities u, v, w to the components α, β, γ of magnetic force (see *Maxwell*, § 616).

Also the magnetic potential of a magnetic shell, and the velocity potential due to a doublet sheet are essentially single valued functions, since the line integral of magnetic force and the circulation are zero for all circuits which do not cut the shell or doublet sheet, and which it is not permissible to cross; on the other hand the magnetic potential due to an electric current, and the velocity potential due to a vortex, although represented by the same quantities, are cyclic functions, the cyclic constant being equal to $2m$, where m is the strength of the vortex. This cyclic constant is equal to the line integral $\int d\phi/ds \,.\, ds$ taken once round a closed circuit embracing the vortex or current once; and in the former case it represents the circulation, and in the latter case the work which would have to be done in moving a magnetic pole once round the current in opposition to the magnetic force exercised by the current (*Maxwell*, § 480).

The potential energy of a magnetic shell of strength -1, placed in a magnetic field, the components of whose vector potential are F, G, H is (*Maxwell*, § 423)

$$\int\left(F\frac{dx}{ds} + G\frac{dy}{ds} + H\frac{dz}{ds}\right) ds.$$

The flux through a closed vortex ring is,

$$\iint (lu + mv + nw)\, dS$$

$$= \iint\left\{l\left(\frac{dN}{dy} - \frac{dM}{dz}\right) + m\left(\frac{dL}{dz} - \frac{dN}{dx}\right) + n\left(\frac{dM}{dx} - \frac{dN}{dy}\right)\right\} dS$$

$$= \int\left(L\frac{dx}{ds} + M\frac{dy}{ds} + N\frac{dz}{ds}\right) ds,$$

and this corresponds to the potential energy of the magnetic shell.

The following table shows the connection between the two subjects:

Hydrodynamical Quantities		*Electromagnetic Quantities*	
Name	Symbol	Name	Symbol
Velocity of Liquid	u, v, w	Magnetic Force	α, β, γ
Molecular Rotation	ξ, η, ζ	Electric Current	u, v, w
	L, M, N	Electromagnetic Momentum	F, G, H
Velocity Potential due to Vortex	ϕ	Magnetic Potential of Current	Ω
Vortex Filament		Electric Current	
Doublet Sheet		Magnetic Shell	
Circulation	κ	Work done in moving a Magnetic Pole once round Current	
Flux through Vortex		Potential Energy of Magnetic Shell	

In addition to the papers cited in the preceding chapter, we may refer to the following by Sir W. Thomson: "Vortex Atoms," *Phil. Mag.* (4) XXXIV.; "Vortex Statics," *Proc. Roy. Soc. Edin.* 1876; "On Maximum and Minimum Energy in Vortex Motion," *Phil. Mag.* (5) XXIII. p. 529.

The theory of rectilinear and circular vortices will be discussed in the second volume.

EXAMPLES.

1. Liquid is contained in a simply-connected surface S; if ϖ is the impulsive pressure at any point of the liquid due to any arbitrary deformation of S subject to the condition that the enclosed volume is not changed, and ϖ' the impulsive pressure for a different deformation, show that

$$\iint \varpi \frac{d\varpi'}{dn} dS = \iint \varpi' \frac{d\varpi}{dn} dS.$$

2. If a sphere be immersed in a liquid, prove that the kinetic energy of the liquid due to a given deformation of its surface, will be greater when the sphere is fixed than when it is free.

3. If V be the attraction potential of a uniform circular lamina of unit density, in the plane of xy, prove that $\omega dV/dz$ will be the velocity potential of a circular vortex filament coinciding with the boundary of the lamina.

4. The boundaries of a liquid are two fixed concentric cylinders of radii a and c. Prove that if the motion of the liquid is irrotational and in two dimensions, the velocity potential must be equal to $\kappa\theta/2\pi$, where κ is the circulation round any closed circuit which embraces the inner cylinder once only; and that the kinetic energy is equal to $\kappa^2 (4\pi)^{-1} \log a/c$.

5. Apply the equations of impulsive motion, to show that if liquid be contained within a closed surface, the circulation and the molecular rotation cannot be altered by any impulse applied to the boundary.

6. A mass of ice is contained within an ellipsoidal case which is rotating in any manner about its centre: prove that if the ice be melted and the boundary be deformed in such a manner that it remains ellipsoidal, the resultant molecular rotation at any point is proportional to the diameter of the ellipsoid which is parallel to the tangent to the vortex line at that point.

CHAPTER V.

ON THE MOTION OF A LIQUID IN TWO DIMENSIONS.

96. THE solution of questions relating to the motion of a liquid in two dimensions, can be most conveniently effected by means of Earnshaw's current function ψ. This function when the motion is irrotational, which will be the case in most of the problems discussed in the present chapter, satisfies the equation

$$\frac{d^2\psi}{dx^2} + \frac{d^2\psi}{dy^2} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

the solution of which is

$$\psi = f(x + \iota y) + F(x - \iota y) \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Also
$$u = \frac{d\psi}{dy}, \quad v = -\frac{d\psi}{dx} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

If the liquid is bounded by fixed surfaces, the normal component of the velocity must vanish at the boundaries. This condition requires that $\psi = \text{const.}$ at all points of boundaries which are fixed.

97. When the cylindrical boundary is in motion, the following conditions must be satisfied at its surface.

(i) Let the cylinder be moving with velocity U parallel to the axis of x, and let θ be the angle which the normal to the cylinder makes with this axis; then at the surface

$$u \cos\theta + v \sin\theta = U \cos\theta.$$

Now $\cos\theta = dy/ds$; $\sin\theta = -dx/ds$; therefore by (3)

$$\frac{d\psi}{ds} = U\frac{dy}{ds}.$$

Integrating along the boundary, we obtain

$$\psi = Uy + A \quad \text{.........................(4)},$$

where A is a constant.

(ii) If the cylinder be moving with velocity V parallel to the axis of y, the surface condition in the same manner can be shewn to be

$$\psi = -Vx + B \quad \text{.........................(5)}.$$

(iii) Let the cylinder be rotating with angular velocity ω; then at the surface

$$u\cos\theta + v\sin\theta = -\omega y\cos\theta + \omega x\sin\theta$$

or
$$\frac{d\psi}{ds} = -\omega r\frac{dr}{ds}.$$

Therefore
$$\psi = -\tfrac{1}{2}\omega r^2 + C \quad \text{.....................(6)},$$

where $r = \sqrt{x^2 + y^2}$.

When there are any number of moving cylinders in the liquid, conditions (4), (5) and (6) must be satisfied at the surfaces of each of the moving cylinders.

In addition to the surface conditions, ψ must satisfy the following conditions at every point of space occupied by the liquid; viz. ψ must be a function which is a solution of Laplace's Equation (1), and which together with its first derivatives must be finite and continuous at every point of the liquid. If the liquid extends to infinity and is at rest there, the first derivatives must vanish at infinity.

Conjugate Functions.

98. Def. *If ξ and η are functions of x and y such that*

$$\xi + \iota\eta = f(x + \iota y) \quad \text{.........................(7)}$$

then ξ and η are called conjugate functions of x and y.

Differentiate (7) with respect to x and y respectively, eliminate the function f, and equate the real and imaginary parts in the resulting equation, and we shall obtain

$$\frac{d\xi}{dx} = \frac{d\eta}{dy};\quad \frac{d\xi}{dy} = -\frac{d\eta}{dx} \quad \text{.....................(8)}.$$

Now, if ϕ and ψ be the velocity potential and current function of a liquid, it follows that if ϕ and ψ are written for ξ and η respectively, equations (8) are satisfied; hence ϕ and ψ are conjugate functions of x and y.

99. Again

$$\frac{d\xi}{dx}\cdot\frac{d\eta}{dx}+\frac{d\xi}{dy}\cdot\frac{d\eta}{dy}=0 \quad \text{.....................(9)},$$

$$\left(\frac{d\xi}{dx}\right)^2+\left(\frac{d\xi}{dy}\right)^2=\left(\frac{d\eta}{dx}\right)^2+\left(\frac{d\eta}{dy}\right)^2=J^2 \quad \text{.........(10)},$$

$$\nabla^2\xi=0,\ \nabla^2\eta=0 \quad \text{.....................(11)},$$

where $$\nabla^2=d^2/dx^2+d^2/dy^2.$$

Equation (9) shows that the curves $\xi=\text{const.}$, $\eta=\text{const.}$ form an orthogonal system. Equations (2), (7) and (11) show that

$$\left.\begin{aligned}2\xi&=f(x+\iota y)+F(x-\iota y)\\ 2\iota\eta&=f(x+\iota y)-F(x-\iota y)\end{aligned}\right\} \quad \text{...............(12)},$$

whence $$\xi-\iota\eta=F(x-\iota y).$$

Hence if $\phi(x, y, c)=0$ be the equation of any family of curves which can be expressed in the form

$$2\chi(c)=2\xi=f(x+\iota y)+F(x-\iota y)$$

the equation of the orthogonal system of curves will be

$$2\iota\eta=f(x+\iota y)-F(x-\iota y),$$

where η is constant along each curve of the orthogonal system.

Again we have

$$d\xi=\frac{d\xi}{dx}dx+\frac{d\xi}{dy}dy,$$

$$d\eta=\frac{d\eta}{dx}dx+\frac{d\eta}{dy}dy.$$

Therefore if ds be the distance between two adjacent points,

$$J^2ds^2=d\xi^2+d\eta^2.$$

Hence if ds_ξ, ds_η be small arcs of the curves ξ and η respectively

$$\left.\begin{aligned}Jds_\xi&=d\eta\\ Jds_\eta&=d\xi\end{aligned}\right\} \quad \text{.......................(13)}.$$

100. If ϕ and ψ are conjugate functions of ξ and η, then ϕ and ψ are conjugate functions of x and y.

For $$\phi + \iota\psi = F(\xi + \iota\eta)$$
and $$\xi + \iota\eta = f(x + \iota y),$$
therefore $$\phi + \iota\psi = \chi(x + \iota y).$$

101. Let p and q be the velocities perpendicular to ξ and η in the directions in which these quantities increase, then

$$\left.\begin{aligned} p &= \frac{d\phi}{ds_\eta} = J\frac{d\phi}{d\xi} = J\frac{d\psi}{d\eta} \\ q &= \frac{d\phi}{ds_\xi} = J\frac{d\phi}{d\eta} = -J\frac{d\psi}{d\xi} \end{aligned}\right\} \quad \ldots\ldots\ldots\ \ldots\ldots (14).$$

If we consider a small curvilinear rectangle bounded by the curves ξ, η; $\xi + \delta\xi$, $\eta + \delta\eta$, the difference between the fluxes over the faces $\xi + \delta\xi$ and $\eta + \delta\eta$, and those over the faces ξ and η is

$$\left(\frac{d^2\phi}{d\xi^2} + \frac{d^2\phi}{d\eta^2}\right) d\xi d\eta = J^2\left(\frac{d^2\phi}{d\xi^2} + \frac{d^2\phi}{d\eta^2}\right) dxdy,$$

but if we choose the two tangents to the curves ξ and η at their point of intersection as the axes of x and y, the difference between these fluxes will be

$$\nabla^2\phi dxdy.$$

Hence $$\nabla^2\phi = J^2\left(\frac{d^2\phi}{d\xi^2} + \frac{d^2\phi}{d\eta^2}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (15).$$

In the case of an irrotationally moving liquid, both sides of this equation must be zero; hence Laplace's equation when transformed into any variables ξ, η which are conjugate functions of x and y, becomes

$$\frac{d^2\phi}{d\xi^2} + \frac{d^2\phi}{d\eta^2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16).$$

If we assume as the value of ψ any solution of (1) or (16) and substitute this value in any of the three equations (4), (5) or (6), we shall obtain a system of curves, any one of which would, by its motion in the prescribed manner, produce lines of flow determined by the equation $\psi = \text{const.}$

102. We shall now give some examples.

(i) Let $$\psi = -\tfrac{1}{2}Va^2\left(\frac{1}{x + \iota y} + \frac{1}{x - \iota y}\right)$$

$$= -\frac{Va^2x}{r^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (17).$$

When $r = a$, $\psi = -Vx$; also the velocity is finite and continuous at all points outside the cylinder $r = a$, and vanishes at infinity; hence ψ is the current function when a circular cylinder of radius a is moving in an infinite liquid with velocity V parallel to y.

The velocity potential is

$$2\iota\phi = Va^2\left(\frac{1}{x+\iota y} - \frac{1}{x-\iota y}\right)$$

or
$$\phi = -\frac{Va^2y}{r^2} \quad \text{..................................(18).}$$

The paths of individual particles of liquid due to the motion of a cylinder along a straight line, have been calculated and traced by Clerk-Maxwell[1].

(ii) If the liquid instead of extending to infinity is bounded by a fixed concentric cylinder of radius c, the *initial* motion of the liquid can be obtained as follows.

Since $(x \pm \iota y)^n$ is a solution of Laplace's equation, it follows that $r^n(A \cos n\theta + B \sin n\theta)$ is also a solution, where n is any quantity positive, negative, real or complex.

Hence if the inner cylinder be moved along the axis of x with initial velocity U, we may put

$$\phi = \left(Ar + \frac{B}{r}\right)\cos\theta.$$

When $r = a$, $d\phi/dr = U\cos\theta$, whence

$$A - \frac{B}{a^2} = U.$$

When $r = c$, $d\phi/dr = 0$, whence

$$A - \frac{B}{c^2} = 0.$$

Therefore
$$\phi = -\frac{Ua^2}{c^2 - a^2}\left(r + \frac{c^2}{r}\right)\cos\theta.$$

This is the expression for the initial value of the velocity potential. The motion at any subsequent time after the cylinders have ceased to be concentric will be determined in § 122.

[1] "On the Displacement in a case of Fluid Motion," *Proc. Lond. Math. Soc.* vol. III. p. 82.

(iii) Let $\psi = \frac{1}{2} A \{(x + \iota y)^3 + (x - \iota y)^3\}$

$$= A(x^3 - 3xy^2) = Ar^3 \cos 3\theta.$$

Substituting in (6) the equation of the boundary becomes,

$$A(x^3 - 3xy^2) + \tfrac{1}{2}\omega(x^2 + y^2) = C \quad \text{............(19)}.$$

If we choose the constants so that the straight line $x = a$, may form part of the boundary, we find

$$A = \frac{\omega}{6a}\,; \quad C = \frac{2\omega a^2}{3}.$$

Hence (19) splits up into the factors

$$(x - a)\,;\; x + y\sqrt{3} + 2a\,;\; x - y\sqrt{3} + 2a.$$

The boundary therefore consists of three straight lines forming an equilateral triangle, whose centre is the origin.

Hence ψ is the current function due to liquid contained in an equilateral prism, which is rotating with angular velocity ω about an axis through the centre of inertia of its cross section. The values of ψ and ϕ, when cleared of imaginaries, are

$$\psi = \frac{\omega}{6a} r^3 \cos 3\theta, \qquad \phi = \frac{\omega}{6a} r^3 \sin 3\theta.$$

(iv) Let $\psi = \frac{1}{2} A \{(x + \iota y)^2 + (x - \iota y)^2\}$

$$= A(x^2 - y^2).$$

Substituting in (6) we find

$$A(x^2 - y^2) + \tfrac{1}{2}\omega(x^2 + y^2) = C \quad \text{...............(20)}.$$

Putting $\dfrac{\omega + 2A}{2C} = \dfrac{1}{a^2}\,;\; \dfrac{\omega - 2A}{2C} = \dfrac{1}{b^2}$,

the equation of the boundary becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{..........................(21)},$$

and $$\psi = -\frac{\omega}{2}\frac{a^2 - b^2}{a^2 + b^2}(x^2 - y^2) \quad \text{..................(22)},$$

ψ is therefore the current function due to the motion of liquid contained in an elliptic cylinder, which is rotating about its axis.

The preceding value of ψ is also the current function, when the liquid is bounded by two concentric, similar and similarly situated elliptic cylinders.

103. *To find the current function when liquid is contained in a rectangular prism which is rotating with angular velocity ω about its axis*[1].

If $2a$, $2c$ be the sides of the cross section of the prism, the boundary conditions are

$$u = \frac{d\psi}{dy} = -\omega y, \qquad \text{when} \quad x = \pm a,$$

$$v = -\frac{d\psi}{dx} = \omega x, \qquad \text{when} \quad y = \pm c.$$

Also $$\nabla^2\psi = 0.$$

Let $$\chi - \tfrac{1}{2}\omega(x^2 + y^2) = \psi;$$

then $$\left.\begin{aligned} \frac{d\chi}{dy} &= 0, \qquad x = \pm a \\ \frac{d\chi}{dx} &= 0, \qquad y = \pm c \end{aligned}\right\} \text{..................(23)},$$

and $$\nabla^2\chi - 2\omega = 0 \text{ ....(24)}.$$

Let $$\chi = \Sigma(\theta \cos\lambda x + \zeta \sin\lambda x),$$

where θ and ζ are functions of y alone. Substituting in the first of (23) we obtain,

$$\Sigma\left(\frac{d\theta}{dy}\cos\lambda a \pm \frac{d\zeta}{dy}\sin\lambda a\right) = 0,$$

therefore $$\zeta = 0,$$

$$\lambda = (2n+1)\frac{\pi}{2a}.$$

Hence $$\chi = \Sigma\theta_{2n+1}\cos(2n+1)\frac{\pi x}{2a} \text{ ....(25)}.$$

Substituting this value of χ in (24), we obtain

$$\Sigma\left\{\frac{d^2}{dy^2} - (2n+1)^2\frac{\pi^2}{4a^2}\right\}\theta_{2n+1}\cos(2n+1)\frac{\pi x}{2a} - 2\omega = 0 \text{ ...(26)}.$$

Now $$\int_{-a}^{a}\cos(2n+1)\frac{\pi x}{2a}\,dx = \frac{(-)^n 4a}{(2n+1)\pi},$$

and $$\int_{-a}^{a}\cos(2m+1)\frac{\pi x}{2a}\cos(2n+1)\frac{\pi x}{2a}\,dx = 0 \text{ or } a,$$

according as m is unequal or equal to n.

[1] Stokes, "On some cases of Fluid Motion," *Trans. Camb. Phil. Soc.* vol. VIII. p. 105. Ferrers, "Solution of certain questions in Potentials and Motion of Liquids," *Quart. Journ.* vol. XV. p. 83. For the expressions for the component velocities of the liquid in terms of elliptic functions, see Greenhill, *Quart. Journ.* vol. XV. p. 144.

Multiplying (26) by $\cos(2n+1)\pi x/2a$, and integrating between the limits a and $-a$, we obtain

$$\frac{d^2\theta_{2n+1}}{dy^2}-\frac{(2n+1)^2\pi^2}{4a^2}\theta_{2n+1}-\frac{(-)^n 8\omega}{(2n+1)\pi}=0,$$

therefore

$$\theta_{2n+1}=A_{2n+1}\cosh(2n+1)\frac{\pi y}{2a}+B_{2n+1}\sinh(2n+1)\frac{\pi y}{2a}-\frac{(-)^n 32a^2\omega}{(2n+1)^3\pi^3}.$$

If we substitute this value of θ_{2n+1} in (25), and then substitute the resulting value of χ in the second of (23), we obtain

$$B_{2n+1}=0,$$

$$A_{2n+1}=\frac{(-)^n 32a^2\omega}{(2n+1)^3\pi^3\cosh(2n+1)\pi c/2a},$$

whence

$$\chi=\frac{32a^2\omega}{\pi^3}\Sigma_0^\infty\frac{(-)^n\cos(2n+1)\pi x/2a\,.\cosh(2n+1)\pi y/2a}{(2n+1)^3\cosh(2n+1)\pi c/2a}$$

$$-\frac{32a^2\omega}{\pi^3}\Sigma_0^\infty\frac{(-)^n\cos(2n+1)\pi x/2a}{(2n+1)^3}.$$

Now if z lies between $\frac{1}{2}\pi$ and $-\frac{1}{2}\pi$,

$$\cos z-\frac{1}{3^3}\cos 3z+\frac{1}{5^3}\cos 5z-\ldots\ldots=\frac{\pi^3}{32}-\frac{\pi z^2}{8};$$

hence the second series is equal to $\omega(a^2-x^2)$, and the value of ψ is therefore

$$\psi=-\omega a^2-\tfrac{1}{2}\omega(x^2-y^2)$$

$$+\frac{32a^2\omega}{\pi^3}\Sigma_0^\infty\frac{(-)^n\cos(2n+1)\pi x/2a\cosh(2n+1)\pi y/2a}{(2n+1)^3\cosh(2n+1)\pi c/2a}.$$

A more symmetrical expression may be obtained from the consideration that ψ must be unaltered when a and x are written for c and y; making these changes and adding the results we obtain,

$$\psi=-\tfrac{1}{2}\omega(a^2+c^2)$$

$$+\frac{16a^2\omega}{\pi^3}\Sigma_0^\infty\frac{(-)^n\cos(2n+1)\pi x/2a\cosh(2n+1)\pi y/2a}{(2n+1)^3\cosh(2n+1)\pi c/2a}$$

$$+\frac{16c^2\omega}{\pi^3}\Sigma_0^\infty\frac{(-)^n\cos(2n+1)\pi y/2c\cosh(2n+1)\pi x/2c}{(2n+1)^3\cosh(2n+1)\pi a/2c}.$$

104. *To find the velocity potential when liquid is contained in a cylinder whose cross section is the sector of a circle, which is rotating about an axis through the centre of the circle*[1].

Let α be the angle of the sector, a the radius of the cylinder, ω its angular velocity, then

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} + \frac{1}{r^2}\frac{d\phi}{d\theta} = 0 \quad \text{..................(27)},$$

and the surface conditions are

$$\frac{1}{r}\frac{d\phi}{d\theta} = \omega r, \quad \text{when } \theta = 0 \text{ or } \alpha \quad \text{..............(28)},$$

$$\frac{d\phi}{dr} = 0, \quad \text{when } r = a \quad \text{..................(29)},$$

also ϕ must not become infinite when $r = 0$.

Let $$\phi = R\cos\lambda(\theta + \beta),$$

where R is a function of r alone. Substituting in (27) we obtain

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \frac{R\lambda^2}{r^2} = 0,$$

the solution of which is

$$R = Ar^\lambda + Br^{-\lambda}.$$

Hence since λ has not at present been determined, the value of ϕ may be written

$$\phi = A_0 r^2 \cos 2(\theta + \beta_0) + \Sigma(Ar^\lambda + Br^{-\lambda})\cos\lambda(\theta + \beta).$$

Substituting in (28) we obtain

$$2A_0 r^2 \sin 2(\theta + \beta_0) + \Sigma\lambda(Ar^\lambda + Br^{-\lambda})\sin\lambda(\theta + \beta) = -\omega r^2.$$

This equation is satisfied, provided

$$\left.\begin{aligned} 2A_0 \sin(2\theta + 2\beta_0) &= -\omega, \\ \Sigma\lambda(Ar^\lambda + Br^{-\lambda})\sin\lambda(\theta + \beta) &= 0, \end{aligned}\right\} \text{ when } \theta = \alpha \text{ or } 0,$$

which requires that

$$2\beta_0 = \tfrac{1}{2}\pi - \alpha, \qquad 2A_0\cos\alpha = -\omega,$$
$$\beta = 0, \qquad \lambda = (2n+1)\pi/\alpha.$$

[1] Stokes, "On the critical values of the sums of periodic series," *Trans. Camb. Phil. Soc.* vol. VIII. p. 533. Greenhill, "Fluid motion in a rotating semi-circular cylinder," *Mess. Math.* vol. VIII. p. 42; "Fluid motion in a rotating quadrantal cylinder," *Ibid.* p. 89; "Fluid motion in a rotating rectangle formed by two concentric circular arcs and two radii," *Ibid.* vol. IX. p. 35; "On the motion of a frictionless liquid in a rotating sector," *Ibid.* vol. X. p. 83.

Therefore

$$\phi = \frac{\omega r^2}{2\cos\alpha}\sin(2\theta - \alpha)$$
$$+ \Sigma_0^\infty \{A_n r^{(2n+1)\pi/\alpha} + B_n r^{-(2n+1)\pi/\alpha}\} \cos(2n+1)\pi\theta/\alpha.$$

Since ϕ must not be infinite when $r = 0$, $B_n = 0$; substituting in (29), we find that for all values of θ between α and 0,

$$\omega a \sec\alpha \sin(2\theta - \alpha) + \frac{\pi}{\alpha}\Sigma_0^\infty A_n (2n+1)\, a^{(2n+1)\pi/\alpha}\cos(2n+1)\pi\theta/\alpha = 0,$$

whence by Fourier's theorem

$$\tfrac{1}{2}\pi A_n (2n+1) a^{(2n+1)\pi/\alpha} = -\omega a^2 \sec\alpha \int_0^\alpha \sin(2\theta - \alpha)\cos(2n+1)\pi\theta/\alpha\, d\theta$$
$$= -\frac{4\omega a^2 \alpha^2}{4\alpha^2 - (2n+1)^2\pi^2},$$

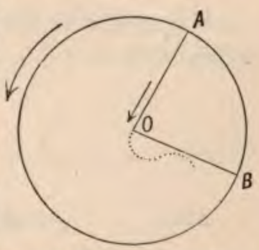
therefore $A_n = \dfrac{8\omega a^2 \alpha}{\pi(2n+1)\{(2n+1)^2\pi^2 - 4\alpha^2\}}\, a^{-(2n+1)\pi/\alpha}$,

and $\phi = \dfrac{\omega}{2\cos\alpha} r^2 \sin(2\theta - \alpha)$

$$+ 8\omega a^2 \alpha\, \Sigma_0^\infty \left(\frac{r}{a}\right)^{(2n+1)\pi/\alpha} \frac{\cos(2n+1)\pi\theta/\alpha}{\pi(2n+1)\{(2n+1)^2\pi^2 - 4\alpha^2\}}.$$

105. The interpretation of this expression presents no difficulty so long as $\alpha < \pi$, but when $\alpha > \pi$ the velocity becomes infinite at the origin. The following explanation of the motion which takes place when this is the case, is given by Prof. Stokes:

"Let OAB be a section of the sector made by a plane perpendicular to the axis, and cutting it in O. Suppose the cylinder turning round O in the direction indicated by the arrow. Then the liquid in contact with OA and near O, will be flowing relatively to OA, towards O, as indicated by the arrow at O. When it gets to O, it will shoot past the face OB; so that there will be formed a surface of discontinuity indicated by the dotted line, extending some way into the liquid, the liquid underneath this line and near O flowing in the direction AO, while the liquid above is nearly at rest."



Whenever a liquid is flowing past a sharp edge, the analytical expression for the velocity, calculated on the assumption that the liquid is perfect and flows according to the electrical law of flow, always becomes infinite at the edge; a result analogous to that which occurs in the theory of the distribution of electricity on conductors, where it is found that the analytical expression for the density upon a conductor having a sharp edge becomes infinite at the edge.

The mathematical investigation of the discontinuous motion which takes place in such cases is one of great difficulty, but certain special cases will be considered in the next Chapter.

106. The problem of finding the velocity potential and current function, when a cylinder whose cross section is a given curve, is moving in an infinite liquid, has been solved in comparatively few cases. The theory of conjugate functions affords a powerful method of attacking such problems, but the principal difficulty consists in finding a relation between the complexes $\xi + \iota\eta$ and $x + \iota y$, such that the given boundary shall be represented by some particular value of one of the functions ξ or η.

The principal solutions of this problem, which have hitherto been obtained, will be given in the following articles.

107. Let
$$x + \iota y = c \cos(\xi - \iota\eta) \quad\text{.................. (30)},$$
then
$$x = c \cos\xi \cosh\eta,$$
$$y = c \sin\xi \sinh\eta,$$
and the curves $\eta = \text{const.}$, $\xi = \text{const.}$ are a family of confocal ellipses and hyperbolas.

If a and b be the semi-axes of the cross section of the ellipse $\eta = \beta$, then
$$a = c \cosh\beta,$$
$$b = c \sinh\beta,$$
$$a^2 - b^2 = c^2.$$

Also
$$J^2 = \frac{2}{c^2(\cosh 2\eta - \cos 2\xi)} \quad\text{................. (31)}.$$

Here η may have any *positive* value, and ξ may have any real value whatever; when $\eta = 0$, the ellipse becomes a double line joining the foci; and when $\eta = \infty$ the curves become circles; also J vanishes at infinity.

Now ψ satisfies the equation

$$\frac{d^2\psi}{d\xi^2} + \frac{d^2\psi}{d\eta^2} = 0 \qquad\qquad (32),$$

and this equation is satisfied by the expression

$$A\eta + \Sigma_1^\infty \epsilon^{-n\eta}(A_n \cos n\xi + B_n \sin n\xi) \qquad\qquad (33),$$

which is the proper form of a potential function outside an elliptic cylinder, since by (14) and (31), it makes the velocity vanish at infinity.

To find the form of ψ inside the cylinder, we observe that (32) is also satisfied by the series

$$\Sigma_1^\infty (A_n \cosh n\eta \cos n\xi + B_n \sinh n\eta \sin n\xi + C_n \sinh n\eta \cos n\xi + D_n \cosh n\eta \sin n\xi) \qquad (34).$$

Now if we examine the components of the velocity in the neighbourhood of the line joining the foci, it will be found that they will be discontinuous, unless $d\psi/d\eta$ and $d\psi/d\xi$ either vanish or change sign in passing from one side of this line to the other; the last two terms of (34) are therefore inadmissible. Hence every potential function, which together with its first derivatives is finite and continuous inside an elliptic cylinder, must be of the form

$$\Sigma_1^\infty (A_n \cosh n\eta \cos n\xi + B_n \sinh n\eta \sin n\xi) \qquad\qquad (35).$$

This value also makes the component velocities finite at the foci; for in the neighbourhood of these points $Jc = (\delta\eta^2 + \delta\xi^2)^{-\frac{1}{2}}$, and from (35) both $d\psi/d\xi$ and $d\psi/d\eta$ are infinitesimals of the first order.

Hence, by (4) and (5) if ψ_x, ψ_y be the current functions when the cylinder $\eta = \beta$ is moving parallel to x and y with velocities U and V respectively,

$$\left.\begin{aligned} \psi_x &= Uc\epsilon^{-\eta+\beta} \sinh\beta \sin\xi \\ \psi_y &= -Vc\epsilon^{-\eta+\beta} \cosh\beta \cos\xi \end{aligned}\right\} \qquad\qquad (36).$$

Again, $\quad r^2 = x^2 + y^2 = \frac{1}{2}c^2(\cosh 2\eta + \cosh 2\xi)$.

Hence, if ψ_3 be the current function when the cylinder is surrounded with liquid and is rotating with angular velocity ω, we must put

$$\psi_3 = A\epsilon^{-2(\eta-\beta)} \cos 2\xi.$$

Substituting in (6) and putting $\eta=\beta$, we obtain

$$A\cos 2\xi+\tfrac{1}{4}c^2\omega(\cosh 2\beta+\cos 2\xi)=C.$$

Hence
$$C=\tfrac{1}{4}c^2\omega\cosh 2\beta,$$
$$A=-\tfrac{1}{4}c^2\omega,$$

and
$$\psi_3=-\tfrac{1}{4}c^2\omega\epsilon^{-2\eta+2\beta}\cos 2\xi \quad\text{..................(37).}$$

The value of the velocity potential may be deduced from the preceding values of ψ or from the corresponding expressions for an ellipsoid, which will be given in Chapter VII. and which were first obtained by Green[1] and Clebsch[2]. The expressions in the text are due to Prof. Lamb[3].

The motion of a liquid in a rotating cylinder, whose cross section is formed (i) by the arcs of confocal ellipses and hyperbolas, (ii) by arcs of confocal parabolas, has been investigated by Dr Ferrers[4].

108. We shall now solve the same problem for a cylinder whose cross section is the inverse of an ellipse with respect to its centre[5].

Let
$$x+\iota y=c\sec(\xi+\iota\eta) \quad\text{....................(38),}$$

then
$$r^4=c^2\left(\frac{x^2}{\cosh^2\eta}+\frac{y^2}{\sinh^2\eta}\right),$$
$$r^4=c^2\left(\frac{x^2}{\cos^2\xi}-\frac{y^2}{\sin^2\xi}\right),$$

whence the curves $\xi=\alpha$, $\eta=\beta$ are the inverses of a family of confocal hyperbolas and ellipses with respect to their common centre.

Also
$$\left.\begin{aligned}\frac{cx}{r^2}&=\cosh\eta\cos\xi,\\ \frac{cy}{r^2}&=\sinh\eta\sin\xi,\\ \frac{2c^2}{r^2}&=\cosh 2\eta+\cos 2\xi,\\ J^2&=\frac{(\cosh 2\eta+\cos 2\xi)^2}{2c^2(\cosh 2\eta-\cos 2\xi)}\end{aligned}\right\}\quad\text{.................(39).}$$

[1] *Trans. Roy. Soc. Edin.* vol. XIII. p. 54.
[2] *Crelle*, vol. LII. p. 119.
[3] "Some hydrodynamical solutions," *Quart. Journ.* XIV. p. 40.
[4] *Quart. Journ.* XVII. p. 227.
[5] *Ibid.* vol. XIX. p. 190, and vol. XXI. p. 336.

Here η may have any *positive* value, and ξ any value positive or negative, but as the values of x and y are periodic with respect to ξ, it is only necessary to consider values of ξ lying between 0 and 2π.

When η is large the curves $\eta = \text{const.}$ consist of small oval curves about the origin, with which they ultimately coincide when $\eta = \infty$; and when $\eta = 0$ they become two double lines extending from the points $x = \pm c$ to infinity in the positive and negative directions respectively.

Also when η is large

$$J = \tfrac{1}{2}\epsilon^{\eta}/c.$$

Hence, within the cylinder, every potential function must be of the form

$$\Sigma_1^\infty \epsilon^{-n\eta}(A_n \cos n\xi + B_n \sin n\xi) \quad\text{...............(40).}$$

Outside the cylinder, every potential function must be of the form

$$\Sigma_1^\infty (A_n \cosh n\eta \cos n\xi + B_n \sinh n\eta \sin n\xi) \quad\text{.........(41),}$$

for the velocities will be discontinuous along the two double lines, unless $d\psi/d\xi$ and $d\psi/d\eta$ either vanish or change sign in crossing from one side of these lines to the other, and (41) is the only form which satisfies these conditions. This form also makes the velocity at the points $x = \pm c$ finite.

Now
$$\begin{aligned} x + \iota y &= c \sec(\xi + \iota\eta) \\ &= \frac{2c\epsilon^{-\eta+\iota\xi}}{1+\epsilon^{-2\eta+2\iota\xi}} \\ &= 2c\,(\epsilon^{-\eta+\iota\xi} - \epsilon^{-3\eta+3\iota\xi} + \epsilon^{-5\eta+5\iota\xi} - \&c.); \end{aligned}$$

therefore
$$\begin{aligned} x &= 2c\,\Sigma_0^\infty (-)^n \epsilon^{-(2n+1)\eta} \cos(2n+1)\xi, \\ y &= 2c\,\Sigma_0^\infty (-)^n \epsilon^{-(2n+1)\eta} \sin(2n+1)\xi. \end{aligned}$$

Hence, if ψ_x, ψ_y be the current functions when the cylinder is moving with velocities U and V parallel to x and y respectively,

$$\psi_x = 2Uc\,\Sigma_0^\infty (-)^n \frac{\epsilon^{-(2n+1)\beta} \sinh(2n+1)\eta \sin(2n+1)\xi}{\sinh(2n+1)\beta} \quad\text{...(42),}$$

$$\psi_y = -2Vc\,\Sigma_0^\infty (-)^n \frac{\epsilon^{-(2n+1)\beta} \cosh(2n+1)\eta \cos(2n+1)\xi}{\cosh(2n+1)\beta} \quad\text{...(43),}$$

where β is the value of η at the boundary.

109. The two series (42) and (43) constitute the complete solution of the problem when the motion of the cylinder is one of translation. The results can however be put into a more compact form by means of elliptic functions. To do this, let

$\xi + \iota\eta = u,\ \xi - \iota\eta = v;\ K/\pi = K'/2\beta$, so that $q = \epsilon^{-2\beta}$; then

$$\psi_x = 2U c\iota \Sigma_0^\infty \frac{(-)^n q^{2n+1}}{1-q^{2n+1}} \left\{\cos(2n+1)u - \cos(2n+1)v\right\}$$

$$= \frac{1}{\pi} UKc\iota \left\{\operatorname{cosecam}\left(\frac{2Ku}{\pi} + K\right) - \operatorname{cosecam}\left(\frac{2Kv}{\pi} + K\right)\right\}$$

$$-\tfrac{1}{2} Uc\iota (\sec u - \sec v);$$

therefore

$$\psi_x = \frac{1}{\pi} UKc\iota \left\{\operatorname{cosecam}\left(\frac{2Ku}{\pi} + K\right) - \operatorname{cosecam}\left(\frac{2Kv}{\pi} + K\right)\right\} + Uy.$$

Putting $2K\xi/\pi + K = \theta,\ 2K\eta/\pi = \phi,\ \mathrm{dn}^2\theta = \alpha,\ \mathrm{sn}^2(\phi, k') = \beta$, and clearing of imaginaries, the term in brackets becomes

$$S = -\frac{2\iota(1-\alpha\beta)k^2 \,\mathrm{sn}\,\phi\,\mathrm{cn}\,\phi\,\mathrm{cn}\,\theta\,\mathrm{dn}\,\theta}{(1-\alpha)(1-k'^2\beta) + \alpha\beta(1-\beta)(\alpha-k'^2)},$$

the functions of ϕ being to modulus k'.

The denominator of S

$$= (1-\alpha\beta)(1+\alpha\beta-\alpha-k'^2\beta),$$

therefore
$$S = -\frac{2\iota k^2 \,\mathrm{sn}\,\phi\,\mathrm{cn}\,\phi\,\mathrm{cn}\,\theta\,\mathrm{dn}\,\theta}{\mathrm{dn}^2\phi - \mathrm{dn}^2\theta\,\mathrm{cn}^2\phi}$$

$$= \frac{2\iota k'^2 \,\mathrm{sn}\,\phi\,\mathrm{cn}\,\phi\,\mathrm{sn}\,\chi}{1-\mathrm{sn}^2\chi\,\mathrm{dn}^2\phi},$$

where
$$\chi = 2K\xi/\pi.$$

Hence we finally obtain

$$\psi_x = -\frac{2}{\pi} UKk'^2 c \,.\, \frac{\mathrm{sn}(2K\xi/\pi)\,\mathrm{sn}(K'\eta/\beta)\,\mathrm{cn}(K'\eta/\beta)}{1-\mathrm{sn}^2(2K\xi/\pi)\,\mathrm{dn}^2(K'\eta/\beta)} + Uy \ldots (44),$$

the functions of ξ being to modulus k, and those of η to modulus k'.

Similarly

$$\psi_y = \frac{1}{\pi} VKk'c \left\{\operatorname{secam}\frac{2Ku}{\pi} + \operatorname{secam}\frac{2Kv}{\pi}\right\} - Vx.$$

Putting $2K\xi/\pi = \chi$, $2K\eta/\pi = \phi$, $\text{dn}^2\chi = \alpha$, $\text{sn}^2(\phi, k') = \beta$, and clearing of imaginaries, the term in brackets becomes

$$S = \frac{2k^2(1-\alpha\beta)\,\text{cn}\,\theta\,\text{cn}\,\phi}{(\alpha - k'^2)(1-\beta) + \alpha\beta(1-\alpha)(1-k'^2\beta)}.$$

The denominator

$$= (1-\alpha\beta)\{\alpha - k'^2(1+\alpha\beta-\beta)\},$$

therefore $$S = \frac{2\,\text{cn}\,\chi\,\text{cn}\,\phi}{1-\text{sn}^2\chi\,\text{dn}^2\phi}.$$

Hence we finally obtain

$$\psi_v = \frac{2}{\pi}VKk'c.\frac{\text{cn}\,(2K\xi/\pi)\,\text{cn}\,(K'\eta/\beta)}{1-\text{sn}^2(2K\xi/\pi)\,\text{dn}^2(K'\eta/\beta)} - Vx \;\ldots(45).$$

110. When the cylinder is rotating about its axis with angular velocity ω, the surface condition is

$$\psi_3 = -\tfrac{1}{2}\omega r^2 + C.$$

Now $$1 + \frac{\sinh 2\beta}{\cosh 2\beta + \cos 2\xi} = \frac{\epsilon^{2\beta} + \cos 2\xi}{\cosh 2\beta + \cos 2\xi}$$

$$= \frac{1}{1+\epsilon^{-2(\beta-\iota\xi)}} + \frac{1}{1+\epsilon^{-2(\beta+\iota\xi)}}$$

$$= 2 + 2\Sigma_1^\infty(-)^n\epsilon^{-2n\beta}\cos 2n\xi,$$

therefore

$$\tfrac{1}{2}r^2 = \frac{c^2}{\cosh 2\beta + \cos 2\xi}$$

$$= c^2\,\text{cosech}\,2\beta + 2c^2\,\text{cosech}\,2\beta\,\Sigma_1^\infty(-)^n\epsilon^{-2n\beta}\cos 2n\xi.$$

Therefore

$$\psi_3 = -\omega c^2\,\text{cosech}\,2\beta$$

$$-2\omega c^2\,\text{cosech}\,2\beta\,\Sigma_1^\infty(-)^n\epsilon^{-2n\beta}\frac{\cosh 2n\eta\cos 2n\xi}{\cosh 2n\beta}\;\ldots(46).$$

111. If liquid is contained in a cylindrical cavity bounded by the curve $\eta = \beta$,

$$\psi = -\omega c^2\,\text{cosech}\,2\beta - 2\omega c^2\,\text{cosech}\,2\beta\,\Sigma_1^\infty(-)^n\epsilon^{-2n\eta}\cos 2n\xi$$

$$= -\omega c^2\,\text{cosech}\,2\beta - \omega c^2\,\text{cosech}\,2\beta\left(\frac{\sinh 2\eta}{\cosh 2\beta + \cos 2\xi} - 1\right)$$

$$= -\frac{c^2\omega\,\text{cosech}\,2\beta\sinh 2\eta}{\cosh 2\eta + \cos 2\xi}\;\ldots\ldots\ldots\ldots(47).$$

112. The results of § 109 admit of various interpretations, by means of which we can obtain the solutions of several problems in other branches of physics. Thus the function ψ_y is

(i) The potential without the cylinder, of the induced charge, when the cylinder is placed in a field of uniform electric force parallel to x.

If we invert with respect to the origin, which is equivalent to putting c^2x/r^2 for x, and $x + \iota y = c\cos(\xi - \iota\eta)$, ψ_y is

(ii) The potential of the induced charge within an elliptic cylinder which encloses an electric system whose potential is Vc^2x/r^2.

(iii) ψ_y is the temperature within a solid elliptic cylinder whose boundary is maintained at a temperature $-Vc^2x/r^2$.

113. The equation

$$x + \iota y = 2c\sec^2 \tfrac{1}{2}(\xi + \iota\eta)$$

represents a family of confocal limaçons. The curves $\eta = \text{const.}$ are the inverses with respect to a focus of a family of confocal ellipses, whilst the curves $\xi = \text{const.}$ are the inverses with respect to the same focus of the orthogonal family of confocal hyperbolas. The current functions due to the motion in an infinite liquid of a cylinder whose cross section is the curve $\eta = \beta$, and also of liquid contained in a rotating cylindrical cavity of this form, may be obtained in a similar manner to that employed in §§ 109—111 (see *Quarterly Journal*, Vol. XX. p. 234).

114. Let us now consider the system of curves given by the equation

$$\xi + \iota\eta = \tfrac{1}{2}\log\frac{(x+\iota y)^2 - c^2}{c^2}.$$

This is equivalent to the system

$$(x^2 - y^2 - c^2)^2 + 4x^2y^2 = c^4\epsilon^{4\xi} \quad \text{..............(48)},$$

$$x^2 - y^2 - c^2 = 2xy\cot 2\eta \quad \text{.................(49)}.$$

(48) is the equation of a family of confocal lemniscates, the distance between whose foci is $2c$; and (49) is the equation of a family of rectangular hyperbolas, each of which passes through the foci of the lemniscates and cuts them orthogonally.

It is easily seen that

$$2x = c\{1 + \epsilon^{2(\xi+\iota\eta)}\}^{\frac{1}{2}} + c\{1 + \epsilon^{2(\xi-\iota\eta)}\}^{\frac{1}{2}},$$

$$2\iota y = c\{1 + \epsilon^{2(\xi+\iota\eta)}\}^{\frac{1}{2}} - c\{1 + \epsilon^{2(\xi-\iota\eta)}\}^{\frac{1}{2}},$$

$$r^2 = c^2(1 + 2\epsilon^{2\xi}\cos 2\eta + \epsilon^{4\xi})^{\frac{1}{2}},$$

$$J = \frac{r\epsilon^{-2\xi}}{c^2}$$

$$= \frac{\epsilon^{-\xi}}{c}(1 + 2\epsilon^{-2\xi}\cos 2\eta + \epsilon^{-4\xi})^{\frac{1}{4}}.$$

ξ and η may have any values whatever. At infinity, $\xi = \infty$, $J = 0$; at either of the foci $\xi = -\infty$ and $J = \epsilon^{-2\xi}/c = \infty$. When $\xi = 0$ the curve becomes the lemniscate of Bernoulli ($r^2 = 2c^2\cos 2\theta$); η and $\frac{1}{2}\pi + \eta$ are the angles which the asymptotes of the hyperbola make with the axis of x, and in the first quadrant η varies from 0 to $\frac{1}{2}\pi$.

Hence, for motion parallel to x,

$$\psi_x = -\tfrac{1}{2}Uc\iota\left[\{1 + e^{-2(\xi-2a-\iota\eta)}\}^{\frac{1}{2}} - \{1 + e^{-2(\xi-2a+\iota\eta)}\}^{\frac{1}{2}}\right]\ldots(50),$$

and for motion parallel to y

$$\psi_y = -\tfrac{1}{2}Vc\left[\{1 + c^{-2(\xi-2a-\iota\eta)}\}^{\frac{1}{2}} + \{1 + c^{-2(\xi-2a+\iota\eta)}\}^{\frac{1}{2}}\right]\ldots(51),$$

where α is the value of ξ at the surface.

115. Before dealing with the rotation of the cylinders, we shall make a short digression for the purpose of considering the coefficients of $\cos n\theta$ in the expansion of $(1 + 2c\cos\theta + c^2)^{\frac{1}{2}}$, which we shall denote by L_n, where $c < 1$.

Now

$$(1 + 2c\cos\theta + c^2)^{\frac{1}{2}} = (1 + c\epsilon^{i\theta})^{\frac{1}{2}}(1 + c\epsilon^{-i\theta})^{\frac{1}{2}}$$

$$= (1 + \tfrac{1}{2}c\epsilon^{i\theta} + S_2c^2\epsilon^{2i\theta} + \ldots S_nc^n\epsilon^{-ni\theta} + \ldots)$$

$$\times(1 + \tfrac{1}{2}c\epsilon^{-i\theta} + S_2c^2\epsilon^{-2i\theta} + \ldots S_nc^n\epsilon^{-ni\theta} + \ldots),$$

where $$S_n = \frac{(-)^{n-1}1.3.5\ldots(2n-3)}{2^n.n!};$$

therefore $$L_n = 2c^n\{S_n + \tfrac{1}{2}S_{n+1}c^2 + S_{n+2}S_2c^4 + S_{n+3}S_3c^6 + \ldots\}.$$

The value of L_n, however, may be put into a more convenient form for calculation, for

$$\tfrac{1}{2}\pi L_{n+1} = \int_0^\pi (1 + 2c\cos\theta + c^2)^{\frac{1}{2}}\cos(n+1)\theta d\theta$$

$$= \frac{c}{2(n+1)}\int_0^\pi \frac{\cos n\theta - \cos(n+2)\theta}{(1 + 2c\cos\theta + c^2)^{\frac{1}{2}}}d\theta.$$

Also

$$\tfrac{1}{2}\pi\,(L_n - L_{n+2}) = \int_0^\pi \frac{(1+2c\cos\theta+c^2)\{\cos n\theta - \cos(n+2)\,\theta\}}{(1+2c\cos\theta+c^2)^{\frac{1}{2}}}\,d\theta$$

$$= \frac{\pi\,(1+c^2)\,(n+1)}{c}\,L_{n+1} + c\int_0^\pi \frac{\cos(n-1)\,\theta - \cos(n+3)\,\theta}{(1+2c\cos\theta+c^2)^{\frac{1}{2}}}\,d\theta$$

$$= \frac{\pi\,(1+c^2)\,(n+1)}{c}\,L_{n+1} + \pi\,\{nL_n + (n+2)\,L_{n+2}\};$$

therefore

$$(2n+5)\,L_{n+2} + (2n-1)\,L_n + \frac{2\,(1+c^2)\,(n+1)}{c}\,L_{n+1} = 0 \ldots (52).$$

Also $\quad \tfrac{1}{2}\pi L_0 = \int_0^\pi (1+2c\cos\theta+c^2)^{\frac{1}{2}}\,d\theta$

$$= (1+c)\,E\,(k,\,\tfrac{1}{2}\pi), \text{ where } k = \frac{2\sqrt{c}}{1+c}.$$

Now $$E\,(k) = k'^2\left(F + k\frac{dF}{dk}\right).$$

Also $$F\,(k) = (1+c)\,F\,(c);$$

therefore $$\frac{dF\,(k)}{dk} = \left\{F\,(c) + (1+c)\frac{dF\,(c)}{dc}\right\}\frac{(1+c)^2\sqrt{c}}{1-c}$$

$$= \left\{E\,(c) - (1-c)\,F\,(c)\right\}\frac{(1+c)^2}{(1-c)^2\sqrt{c}};$$

therefore $$E\,(k) = \frac{2E\,(c)}{1+c} - (1-c)\,F\,(c);$$

therefore $$L_0 = \frac{2}{\pi}\,\{2E - (1-c^2)\,F\} \ldots\ldots\ldots\ldots\ldots\ldots (53).$$

Again,

$$\tfrac{1}{2}\pi L_1 = \int_0^\pi (1+2c\cos\theta+c^2)^{\frac{1}{2}}\cos\theta\,d\theta$$

$$= \int_0^\pi \frac{c\sin^2\theta\,d\theta}{(1+2c\cos\theta+c^2)^{\frac{1}{2}}}$$

$$= cF\,(c) - \tfrac{1}{4}\pi L_1 + \tfrac{1}{2}\,(1+c^2)\int_0^\pi \frac{\cos\theta\,d\theta}{(1+2c\cos\theta+c^2)^{\frac{1}{2}}};$$

therefore $$\tfrac{3}{4}\pi L_1 = cF + \frac{1+c^2}{4c}\,\{\tfrac{1}{2}\pi L_0 - (1+c^2)\,F\};$$

therefore $$L_1 = \frac{2}{3\pi c}\,\{(1+c^2)\,E - (1-c^2)\,F\} \ldots\ldots\ldots\ldots (54).$$

Having obtained the values of L_0 and L_1, the values of the successive functions can be calculated by means of the sequence equation (52).

116. To find the current function due to the rotation of the cylinder in an infinite liquid.

(i) Let ξ be positive at the surface of the cylinder and equal to α, then

$$r^2 = c^2\epsilon^{2\alpha}(1 + 2\epsilon^{-2\alpha}\cos 2\eta + \epsilon^{-4\alpha})^{\frac{1}{2}}$$
$$= c^2e^{2\alpha}\Sigma_0^\infty L_n(\alpha)\cos 2n\eta,$$

where $L_n(\alpha)$ is put for $L_n(\epsilon^{-2\alpha})$.

Hence $$\psi_3 = -\tfrac{1}{2}\omega c^2\epsilon^{2\alpha}\Sigma L_n(\alpha)\,\epsilon^{-2n(\xi-\alpha)}\cos 2n\eta \quad\text{.........(55).}$$

(ii) When ξ is negative at the surface, the cylinder consists of two portions, which we must suppose to be rigidly connected together; in this case let $\xi = -\alpha$ at the surface, where α is positive; then

$$\psi_3 = -\tfrac{1}{2}\omega c^2\Sigma_0^\infty L_n(\alpha)\,\epsilon^{-2n(\xi+\alpha)}\cos 2n\eta \quad\text{............ (56).}$$

In the case of a cylindrical cavity filled with liquid, the values of ψ are

$$-\tfrac{1}{2}\omega c^2\epsilon^{2\alpha}\Sigma_1^\infty\epsilon^{2n(\xi-\alpha)}L_n(\alpha)\cos 2n\eta - \tfrac{1}{2}\omega c^2\epsilon^{2\alpha}L_0(\alpha)\text{......(57),}$$

and $$-\tfrac{1}{2}\omega c^2\Sigma_1^\infty\epsilon^{2n(\alpha+\xi)}L_n(\alpha)\cos 2n\eta - \tfrac{1}{2}\omega c^2L_0(\alpha)\text{......... (58).}$$

117. When $\alpha = 0$, and the cross section becomes a lemniscate of Bernoulli, the preceding formulae become much simplified. Putting $u = x + \iota y$, $v = x - \iota y$, we obtain

$$\psi_x = -\tfrac{1}{2}Uc\iota\left\{\frac{v}{\sqrt{v^2-c^2}} - \frac{u}{\sqrt{u^2-c^2}}\right\}\text{............(59),}$$

$$\psi_y = -\tfrac{1}{2}Vc\left\{\frac{v}{\sqrt{v^2-c^2}} + \frac{u}{\sqrt{u^2-c^2}}\right\}\text{............(60).}$$

118. The values of ψ when the cylinder is rotating about its axis may be obtained in this case without having recourse to the general formulae of § 116, for the value of r^2 at the boundary is $2c^2\cos\eta$, whence $\psi_3 = -\omega c^2\epsilon^{-\xi}\cos\eta$. This may be expressed in the form

$$\psi_3 = -\tfrac{1}{2}\omega c^3\left\{\frac{1}{\sqrt{u^2-c^2}} + \frac{1}{\sqrt{v^2-c^2}}\right\}\text{............(61).}$$

119. To find ψ when the liquid is contained in a cylindrical cavity formed by one of the loops of the curve, we observe that ψ cannot contain any lower power of ϵ^{ξ} than $\epsilon^{2\xi}$ (ξ being of course negative), otherwise the velocities would be infinite at the foci, where $J = \epsilon^{-2\xi}/c$. Now

$$r^2 = 2c^2 \cos \eta\,;$$

also for all values of η between $\frac{1}{2}\pi$ and $-\frac{1}{2}\pi$ both exclusive,

$$\cos \eta = \tfrac{1}{2}\pi + \sum_{n=1}^{n=\infty} \frac{(-)^{n-1} \cos (2n+1)\,\eta}{2n+1}.$$

Therefore

$$\psi = -\,\omega c^2 \left\{\tfrac{1}{4}\pi + \Sigma_1^{\infty} \frac{(-)^{n-1}\, \epsilon^{(2n+1)\xi} \cos (2n+1)\,\eta}{2n+1}\right\}$$

$$= -\,\omega c^2 \left(\tfrac{1}{4}\pi + \epsilon^{\xi} \cos \eta + \tfrac{1}{2} \tan^{-1} \frac{\cos \eta}{\sinh \xi}\right) \text{ .... } (62).$$

120. Lastly, let us consider the equation

$$x + \iota y = c \tan \tfrac{1}{2}\,(\xi + \iota\eta) \text{ .... } (63).$$

Then

$$\tan \xi = \tan \tfrac{1}{2}\,(\xi + \iota\eta + \xi - \iota\eta)$$

$$= \frac{2cx}{c^2 - x^2 - y^2}.$$

Therefore

$$x^2 + y^2 + 2cx \cot \xi - c^2 = 0 \text{ .... } (64).$$

Also

$$\iota \tanh \eta = \tan \tfrac{1}{2}\,(\xi + \iota\eta - \xi + \iota\eta)$$

$$= \frac{2c\iota y}{c^2 + x^2 + y^2}.$$

Therefore

$$x^2 + y^2 - 2cy \coth \eta + c^2 = 0 \text{ .... } (65).$$

Again,

$$x + \iota y = c\,\frac{\sin \frac{1}{2}\,(\xi + \iota\eta) \cos \frac{1}{2}\,(\xi - \iota\eta)}{\cos \frac{1}{2}\,(\xi + \iota\eta) \cos \frac{1}{2}\,(\xi - \iota\eta)}$$

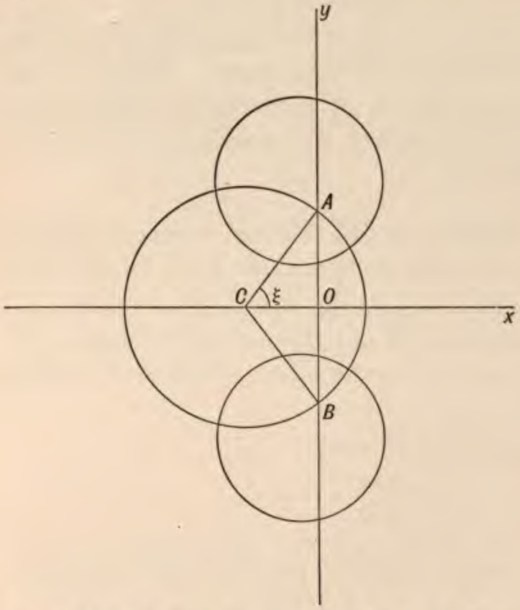
$$= c\,\frac{\sin \xi + \iota \sinh \eta}{\cosh \eta + \cos \xi}.$$

Therefore

$$\left.\begin{aligned} x &= \frac{c \sin \xi}{\cosh \eta + \cos \xi} \\ y &= \frac{c \sinh \eta}{\cosh \eta + \cos \xi} \end{aligned}\right\} \text{ .... } (66),$$

$$J = \frac{1}{c}\,(\cosh \eta + \cos \xi).$$

121. Equation (64) represents a family of circles whose centres lie on the axis of x, at a distance $-c\cot\xi$ from the origin O, and whose radii are equal to $c\operatorname{cosec}\xi$. Each circle passes through two fixed points A and B on the axis of y, whose distances from O are c and $-c$.



The angle ξ is half the angle subtended by AB at the centre of the circle. Hence the curve $\xi = 0$ represents the portion AB of the axis of y. When ξ has any positive value between 0 and π the curve consists of that segment of a circle passing through A and B which lies on the positive side of the axis of y; and when $\xi = \pi$ the curve becomes the whole of the axis y except the portion AB.

When ξ has any negative value between 0 and $= \pi$ the curves consist of segments of circles described on AB, and which lie on the negative side of the axis of y.

Equation (65) represents two families of circles whose centres lie on the axis of y, at distances $\pm c\coth\eta$ from 0, and whose radii are equal to $c\operatorname{cosech}\eta$. These circles do not cut the axis of x.

When $\eta = \infty$ the curve reduces to the point A; when η has any positive value the curve represents a circle surrounding this point; and when $\eta = 0$ the curve becomes the axis of x. When η has any negative value the curve represents a circle surrounding the point B, with which it ultimately coincides, when $\eta = -\infty$.

Let P be any point on one of the circles A, then

$$AP^2 = x^2 + (y-c)^2$$
$$= 2cy(\coth\eta - 1),$$
$$BP^2 = 2cy(\coth\eta + 1);$$
$$AP/BP = \epsilon^{-\eta}.$$

Whence every circle of the system η is such that the ratio AP/BP is constant along each circle; therefore A and B are the common inverse points of each circle of this system. In consequence of this property the coordinates ξ and η are called *dipolar coordinates.*

122. We can now find the current function when two circular cylinders are moving in any manner in an infinite liquid[1].

Let $\eta = \alpha$, $\eta = -\beta$ be the equations of the two cylinders surrounding the points A and B respectively; and let x_1, y_1; $x_2, -y_2$ be coordinates of any point on the cylinders A and B respectively, then

$$x_1 + \iota y_1 = c\tan\tfrac{1}{2}(\xi + \iota\alpha)$$
$$= c\iota\frac{1-\epsilon^{\iota\xi-\alpha}}{1+\epsilon^{\iota\xi-\alpha}}$$
$$= c\iota\{1 + 2\Sigma_1^\infty(-)^n\epsilon^{-n\alpha}(\cos n\xi + \iota\sin n\xi)\};$$

therefore
$$\left.\begin{aligned} x_1 &= -2c\,\Sigma_1^\infty(-)^n\epsilon^{-n\alpha}\sin n\xi \\ y_1 &= c + 2c\,\Sigma_1^\infty(-)^n\epsilon^{-n\alpha}\cos n\xi \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots(67).$$

Again,

$$x_2 - \iota y_2 = c\tan\tfrac{1}{2}(\xi - \iota\beta)$$
$$= \frac{c}{\iota}\frac{1-\epsilon^{-\iota\xi-\beta}}{1+\epsilon^{-\iota\xi-\beta}}$$
$$= \frac{c}{\iota}\left\{1 + 2\Sigma_1^\infty(-)^n\epsilon^{-n\beta}(\cos n\xi - \iota\sin n\xi)\right\};$$

therefore
$$\left.\begin{aligned} x_2 &= -2c\,\Sigma_1^\infty(-)^n\epsilon^{-n\beta}\sin n\xi \\ y_2 &= c + 2c\,\Sigma_1^\infty(-)^n\epsilon^{-n\beta}\cos n\xi \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots(68).$$

Let u, v be the component velocities parallel to x and y of the cylinder A, and u', v' those of B; then

$$\left.\begin{aligned} \psi &= uy_1 - vx_1 + \text{const. at } A \\ \psi &= -u'y_2 - v'x_2 + \text{const. at } B \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots(69).$$

[1] Greenhill, "Functional Images in Cartesians," *Quart. Journ.*, vol. XVIII. pp. 356—362. See also Hicks, *Ibid.* vol. XVI. pp. 113 and 193.

Hence

$$\psi = 2c\,\Sigma_1^\infty\,(-)^n\,\epsilon^{-n\alpha}\,\frac{\sinh n(\eta+\beta)}{\sinh n(\alpha+\beta)}\,(u\cos n\xi + v\sin n\xi)$$
$$-\,2c\,\Sigma_1^\infty\,(-)^n\,\epsilon^{-n\beta}\,\frac{\sinh n(\alpha-\eta)}{\sinh n(\alpha+\beta)}\,(u'\cos n\xi - v'\sin n\xi)\ldots(70).$$

If the cylinder α were moving inside the cylinder β, we should obtain in the same manner

$$\psi = 2c\,\Sigma_1^\infty\,(-)^n\,\epsilon^{-n\alpha}\,\frac{\sinh(\eta-\beta)}{\sinh(\alpha-\beta)}\,(u\cos n\xi + v\sin n\xi)$$
$$+\,2c\,\Sigma_1^\infty\,(-)^n\,\epsilon^{-n\beta}\,\frac{\sinh(\alpha-\eta)}{\sinh(\alpha-\beta)}\,(u'\cos n\xi + v'\sin n\xi)\ \ldots(71).$$

123. We shall hereafter require an expression for the kinetic energy T of an infinite liquid in which two cylinders are moving. By Green's theorem,

$$\frac{2T}{\rho} = \left[\int_{-\pi}^{\pi}\psi\,\frac{d\psi}{d\eta}\,d\xi\right]_{\eta=\alpha} - \left[\int_{-\pi}^{\pi}\psi\,\frac{d\psi}{d\eta}\,d\xi\right]_{\eta=\beta}.$$

Now

$$\psi_\alpha = 2c\,\Sigma_1^\infty\,(-)^n\,\epsilon^{-n\alpha}\,(u\cos n\xi + v\sin n\xi),$$
$$\left(\frac{d\psi}{d\eta}\right)_\alpha = 2c\,\Sigma_1^\infty\,(-)^n\,n\epsilon^{-n\alpha}\coth n(\alpha+\beta)\,(u\cos n\xi + v\sin n\xi)$$
$$+\,2c\,\Sigma_1^\infty\,(-)^n\,n\epsilon^{-n\beta}\,\mathrm{cosech}\,n(\alpha+\beta)\,(u'\cos n\xi - v'\sin n\xi).$$

Hence the first integral

$$= 4\pi c^2(u^2+v^2)\,\Sigma_1^\infty\,n\epsilon^{-2n\alpha}\coth n(\alpha+\beta)$$
$$+\,4\pi c^2(uu'-vv')\,\Sigma_1^\infty\,n\epsilon^{-n(\alpha+\beta)}\,\mathrm{cosech}\,n(\alpha+\beta).$$

Similarly the second integral is equal to

$$-\,4\pi c^2(u'^2+v'^2)\,\Sigma_1^\infty\,n\epsilon^{-2n\beta}\coth n(\alpha+\beta)$$
$$-\,4\pi c^2(uu'-vv')\,\Sigma_1^\infty\,n\epsilon^{-n(\alpha+\beta)}\,\mathrm{cosech}\,n(\alpha+\beta).$$

Hence

$$2T = P(u^2+v^2) + Q(u'^2+v'^2) + 2L(uu'-vv')\ \ldots\ldots(72),$$

where
$$\left.\begin{aligned} P &= 4\pi\rho c^2\,\Sigma_1^\infty\,n\epsilon^{-2n\alpha}\coth n(\alpha+\beta)\\ Q &= 4\pi\rho c^2\,\Sigma_1^\infty\,n\epsilon^{-2n\beta}\coth n(\alpha+\beta)\\ L &= 4\pi\rho c^2\,\Sigma_1^\infty\,n\epsilon^{-n(\alpha+\beta)}\,\mathrm{cosech}\,n(\alpha+\beta)\end{aligned}\right\}\ \ldots\ldots\ldots(73).$$

124. Before we can make use of the foregoing values of P, Q and L, it will be necessary to express them in terms of the radii a and b of the two circles and their coordinates. To do this, let $\theta_1 = \epsilon^{-\alpha}$, $\theta_2 = \epsilon^{-\beta}$, $q = \epsilon^{-\alpha-\beta}$; then

$$P = 4\pi c^2 \Sigma_1^\infty n\theta_1^{2n} \frac{1+q^{2n}}{1-q^{2n}}$$

$$= 4\pi c^2 \sum_{n=1}^{n=\infty} \left\{ n\theta_1^{2n} + 2n \sum_{m=1}^{m=\infty} n\theta^{2n} q^{2mn} \right\}.$$

Now $$q + 2q^2 + 3q^3 + \ldots\ldots = \frac{q}{(1-q)^2};$$

therefore, inverting the order of summation,

$$P = 4\pi c^2 \sum_{m=1}^{m=\infty} \left\{ \frac{\theta_1^2}{(1-\theta_1^2)^2} + \frac{2\theta_1^2 q^{2m}}{(1-\theta_1^2 q^{2m})^2} \right\}.$$

Now $$a = c \operatorname{cosech} \alpha = 2c\theta_1/(1-\theta_1^2);$$

therefore $$P = \rho\pi a^2 \left\{ 1 + 2\Sigma_1^\infty \frac{(1-\theta_1^2)^2 q^{2m}}{(1-\theta_1^2 q^{2m})^2} \right\} \ldots\ldots\ldots\ldots\ldots(74).$$

Similarly $$Q = \rho\pi b^2 \left\{ 1 + 2\Sigma_1^\infty \frac{(1-\theta_2^2)^2 q^{2m}}{(1-\theta_2^2 q^{2m})^2} \right\} \ldots\ldots\ldots\ldots(75.)$$

Again $$L = \rho 8\pi c^2 \Sigma_1^\infty \frac{nq^{2n}}{1-q^{2n}}$$

$$= \rho 8\pi c^2 \sum_{n=1}^{n=\infty} \sum_{m=1}^{m=\infty} nq^{2mn}$$

$$= \rho 8\pi c^2 \sum_{m=1}^{m=\infty} \frac{q^{2m}}{(1-q^{2m})^2}$$

$$= \rho 2\pi ab \sum_{m=1}^{m=\infty} \frac{(1-\theta_1^2)(1-\theta_2^2) q^{2m-1}}{(1-q^{2m})^2} \ldots\ldots\ldots\ldots(76).$$

Since the quantities θ_1, θ_2 are functions of the respective distances of the circles α and β from the axis of y, these values of P, Q and L are of the required form. The coordinate x does not enter into the expressions for the coefficients.

The kinetic energy of a liquid in which two cylinders are moving, was first obtained by Mr W. M. Hicks[1]: the investigation given in the text is due to Prof. Greenhill[2].

[1] "On the motion of two cylinders in a fluid," *Quart. Journ.* vol. XVI. pp. 113 and 193.

[2] "Functional Images in Cartesians," *Quart. Journ.* vol. XVIII. pp. 356—362.

EXAMPLES.

1. An elliptic cylinder is filled with liquid which has molecular rotation ζ at every point, and whose particles move in planes perpendicular to the axis; prove that the lines of flow are similar ellipses described in periodic time

$$\frac{\pi (a^2+b^2)}{ab\zeta}.$$

2. A fixed cylinder whose cross section is any one of the lemniscates $rr'=c^2$, where c is any constant and $2a$ is the distance between the points from which r, r' are measured, is surrounded by an infinite mass of water in steady cyclic irrotational motion; show that the stream lines are all lemniscates of the same system, and that the velocity along a stream line at any point varies as the distance from the centre.

Prove also that the polar coordinates (referred to the centre) of a liquid particle in terms of the time t are given by

$$r^2 = a^2 \mathrm{cn}\mu t \pm c^2 \mathrm{dn}\mu t,$$
$$2\theta = \mathrm{am}\mu t, \ k = a/c.$$

3. The cross section of a cylinder is a sector formed by the circle $r=a$, and the lines $\theta = \pm\alpha$. Prove that if the cylinder be rotating with angular velocity ω,

$$\psi = -\tfrac{1}{2}\omega r^2 \frac{\cos 2\theta}{\cos 2\alpha} - 8\omega a^2 \alpha \Sigma_0^\infty (-)^{n+1} \frac{(r/a)^{(2n+1)\pi/2\alpha} \cos(2n+1)\pi\theta/2\alpha}{(2n+1)\pi\{(2n+1)^2\pi^2 - 16\alpha^2\}}.$$

4. The transverse section of a uniform prismatic vessel is of the form bounded by the two intersecting hyperbolas represented by the equations

$$\sqrt{2}(x^2-y^2)+x^2+y^2=a^2, \ \sqrt{2}(y^2-x^2)+x^2+y^2=b^2.$$

If the vessel be filled with water and made to rotate with angular velocity ω about its axis, prove that the initial component velocities at any point (x, y) of the water will be

$$\frac{\omega}{a^2+b^2}\{2y^3 - 6x^2y + \sqrt{2}(a^2-b^2)y\}$$

$$-\frac{\omega}{a^2+b^2}\{2x^3 - 6xy^2 + \sqrt{2}(b^2-a^2)x\}$$

respectively.

8—2

5. A cylinder whose cross section is the limaçon

$$\frac{r}{2c} = \cos^2 \tfrac{1}{2}\theta \operatorname{sech}^2 \tfrac{1}{2}\beta + \sin^2 \tfrac{1}{2}\theta \operatorname{cosech}^2 \tfrac{1}{2}\beta,$$

is in motion in an infinite liquid with velocities U, V parallel to the lines $\theta = 0$, $\theta = \frac{1}{2}\pi$ respectively; prove that

$$\psi = 8Uc\Sigma_1^\infty (-)^{n-1} n\epsilon^{-n\beta} \operatorname{cosech} n\beta \sinh n\eta \sin n\xi$$
$$- 8Vc\Sigma_1^\infty (-)^{n-1} n\epsilon^{-n\beta} \operatorname{sech} n\beta \cosh n\eta \cos n\xi,$$

where ξ and η are conjugate functions such that

$$x + \iota y = 2c \sec^2 \tfrac{1}{2} (\xi + \iota\eta).$$

6. Prove that if the cylinder in the last example be rotating in an infinite liquid with angular velocity ω,

$$\psi = -8\omega c^2 \operatorname{cosech}^3 \beta \{\cosh\beta + 2\cosh\beta\Sigma_1^\infty (-)^n \epsilon^{-n\beta} \operatorname{sech} n\beta \cosh n\eta \cos n\xi$$
$$+ 2\sinh\beta\Sigma_1^\infty n(-)^n \epsilon^{-n\beta} \operatorname{sech} n\beta \cosh n\eta \cos n\xi\},$$

and that if a cylindrical cavity of this form be filled with liquid and made to rotate,

$$\psi = -\frac{8\omega c^2}{\sinh^3\beta}\left\{\frac{\cosh\beta\sinh\eta}{\cosh\eta + \cos\xi} - \frac{\sinh\beta\,(1 + \cosh\eta\cos\xi)}{(\cosh\eta + \cos\xi)^2}\right\}.$$

7. A circular cylinder is moving with unit velocity parallel to the axis of x; prove that if there is cyclic irrotational motion about the cylinder the velocity potential is

$$\phi = \frac{\kappa\theta}{2\pi} - \frac{a^2 x}{r^2},$$

where κ is the circulation round any closed circuit embracing the cylinder once.

8. A hollow cylinder of radius a, closed at both ends, is divided into two parts by a plane diaphragm through its axis, and filled with liquid. If the vessel be made to rotate about its axis with angular velocity ω, prove that the motion of the liquid relative to the vessel will be such that its velocity potential is

$$\phi = C + \tfrac{1}{2}\omega r^2 \sin 2\theta + \frac{\omega a^2}{8\pi}\left[\left\{\left(\frac{r^2}{a^2} + \frac{a^2}{r^2}\right)\cos 2\theta - 2\right\}\log\frac{r^2 + 2ar\cos\theta + a^2}{r^2 - 2ar\cos\theta + a^2}\right.$$
$$\left. - 2\left(\frac{r^2}{a^2} - \frac{a^2}{r^2}\right)\sin 2\theta \tan^{-1}\frac{2ar\sin\theta}{a^2 - r^2} - 4\left(\frac{r}{a} + \frac{a}{r}\right)\cos\theta\right],$$

where r, θ are polar coordinates of any point of the liquid.

9. Prove that

$$\phi = \log \frac{(x+a)^2 + y^2}{(x-a)^2 + y^2}$$

gives a possible motion in two dimensions. Find the form of the stream lines, and prove that the curves of equal velocity are lemniscates.

10. In the irrotational motion of a liquid, prove that the motion derived from it by turning the direction of motion at each point in one direction through 90° without changing the velocity, will also be a possible irrotational motion, the conditions at the boundaries being altered so as to suit the new motion.

Discuss the motion obtained in this way from the preceding example.

11. Liquid is moving irrotationally in two dimensions, between the space bounded by the two lines $\theta = \pm \frac{1}{6}\pi$ and the curve $r^3 \cos 3\theta = a^3$. The bounding curves being at rest, prove that the velocity potential is of the form

$$\phi = r^3 \sin 3\theta.$$

12. The space between the elliptic cylinder $(x/a)^2 + (y/b)^2 = 1$, and a similarly situated and coaxial cylinder bounded by planes perpendicular to the axis is filled with liquid, and made to rotate with angular velocity ω about a fixed axis. Prove that the velocity potential with reference to the principal axes of the cylinder is $\omega (a^2 - b^2)\, xy/(a^2 + b^2)$, and that the surfaces of equal pressure when the angular velocity is constant, are the hyperbolic cylinders

$$\frac{x^2}{3a^2 + b^2} - \frac{y^2}{3b^2 + a^2} = C.$$

13. If $\phi = f(x, y)$, $\psi = F(x, y)$ are the velocity potential and current function of a liquid, and if we write

$$x = f(\phi, \psi),\ y = F(\phi, \psi)$$

and from these expressions find ϕ and ψ; prove that the new values of ϕ and ψ will be the velocity potential and current function of some other motion of a liquid.

Hence prove that if $\phi = x^2 - y^2$, $\psi = 2xy$, the transformation gives the motion of a liquid in the space bounded by two confocal and coaxial parabolic cylinders.

14. In example 12 prove that the paths of the particles relative to the cylinder are similar ellipses, and that the paths in space are similar to the pericycloid

$$x = (a+b)\cos\theta + (a-b)\cos\left(\frac{a+b}{a-b}\right)^2\theta,$$

$$y = (a+b)\sin\theta + (a-b)\sin\left(\frac{a+b}{a-b}\right)^2\theta.$$

15. Water is enclosed in a vessel bounded by the axis of y and the hyperbola $2(x^2-3y^2)+x+my=0$, and the vessel is set rotating about the axis of z. Prove that

$$\phi = 2(3x^2y - y^3) + xy - \tfrac{1}{2}m(x^2-y^2),$$

$$\psi = 2(x^3 - 3xy^2) + \tfrac{1}{2}(x^2-y^2) + mxy.$$

16. When the stream lines for steady motion are similar concentric and similarly situated ellipses, the motion of a particle is the same as if it were acted upon by a central force to the centre; and if the potential of the impressed forces is a function of the distance from the centre, the lines of equal pressure are circles.

17. The coordinates (x, y) of a particle at time t are given by

$$x = a + A\cos 2n\pi t + B\sin 2n\pi t,$$

$$y = b + \lambda A\sin 2n\pi t - \lambda B\cos 2n\pi t,$$

where A, B, λ and n are constants with regard to x and y, but A and B functions of a and b. Prove that if the different particles corresponding to different values of a and b are the particles of a liquid, A and B must be conjugate functions of the complex $a + \iota b/\lambda$. Under what conditions is a free surface possible?

18. The space between two confocal coaxial elliptic cylinders is filled with liquid which is at rest. Prove that if the outer cylinder be moved with velocity U parallel to the major axis, and the inner with relative velocity V in the same direction, the velocity potential of the initial motion will be

$$\phi = Uc\cosh\eta\cos\xi - Vc\frac{\cosh(\beta-\eta)}{\cosh(\beta-\alpha)}\sinh\alpha\cos\xi,$$

where $\eta=\beta$, $\eta=\alpha$ are the equations of the outer and inner cylinders respectively, and $2c$ the distance between their foci.

19. If in the last example the outer cylinder were to rotate with angular velocity Ω, and the inner with angular velocity ω, prove that initially

$$\phi = \tfrac{1}{4}\Omega c^2 \frac{\cosh 2(\eta - \alpha)}{\sinh 2(\beta - \alpha)} \sin 2\xi - \tfrac{1}{4}\omega c^2 \frac{\cosh 2(\beta - \eta)}{\sinh 2(\beta - \alpha)} \sin 2\xi.$$

20. If $u = x + \iota y$, $v = x - \iota y$, and n be any positive real quantity, prove that when a cylinder whose cross section is the curve $r^n = 2c^n \cos n\theta$ is moving with component velocities U, V parallel to the axes, in an infinite liquid, the current function is

$$\psi = U\psi_x + V\psi_y,$$

where
$$\psi_x = -\tfrac{1}{2} c\iota \{v (v^n - c^n)^{-\frac{1}{n}} - u (u^n - c^n)^{-\frac{1}{n}}\},$$

$$\psi_y = -\tfrac{1}{2} c \{v (v^n - c^n)^{-\frac{1}{n}} + u (u^n - c^n)^{-\frac{1}{n}}\}.$$

Hence prove that if the cross section is the cardioid

$$r = 2c(1 + \cos\theta),$$

$$\psi_x = 2rc^{\frac{3}{2}} \sin \tfrac{1}{2}\theta (\sqrt{r} - \sqrt{c} \cos \tfrac{1}{2}\theta)(r + c - 2\sqrt{rc} \cos \tfrac{1}{2}\theta)^{-2},$$

$$\psi_y = -rc(r + c\cos\theta - 2\sqrt{rc}\cos\tfrac{1}{2}\theta)(r + c - 2\sqrt{rc}\cos\tfrac{1}{2}\theta)^{-2}.$$

CHAPTER VI.

ON DISCONTINUOUS MOTION.

125. In the preceding chapter, we obtained expressions for the velocity potential and the current function of a liquid which is flowing past an elliptic cylinder, and it might be thought that by making the minor axis of the cross section vanish, we could obtain the solution for a stream which is flowing past a rectangular plate. This however is not the case; for if the minor axis be made to vanish, it will be found that the velocity of the liquid becomes infinite at the edges, and therefore the pressure becomes equal to $-\infty$, which indicates that a hollow would be formed in the neighbourhood of the edges. In order that the motion represented by the formulae should be possible, it would be necessary that at every point of the liquid boundary of the hollow, the pressure should be constant, and therefore the liquid boundary would have to be a line of constant pressure as well as a stream line; but it is not difficult to show from the formulae that it is not possible for a line of constant pressure to coincide with a stream line, and hence the formulae fail when the cylinder degenerates into a rectangular plate.

126. The problem of determining the steady motion of heat and electricity, is precisely the same as that of determining the motion of an irrotationally moving liquid subjected to the same boundary conditions, so far as the velocity potential is concerned; but there is an important distinction between the two problems, for in the former the pressure condition does not exist. Hence the solution of problems in the conduction of heat or electricity cannot

receive a hydrodynamical interpretation, unless the value of the pressure given by that solution never becomes negative at any point occupied by the liquid;—in other words, whenever it is possible for the liquid to flow according to the electrical law of flow—; but when this is not the case, the hydrodynamical application of such formulae would give results, which although in many cases approximately representative of the motion at a considerable distance from the region of negative pressure, certainly do not give correct results in the neighbourhood of this region.

127. We have noticed in Chapter IV, that there is nothing in the nature of a perfect fluid to prevent slipping taking place between two contiguous layers, and we have shown that a surface along which slipping takes place is a surface of discontinuity, which possesses the properties of a vortex sheet; but the possibility of such slipping is not taken into account in the ordinary theory, which assumes that the liquid flows according to the electrical law. But in order to solve problems in which liquid is flowing past a sharp edge, it will be necessary to take into consideration the possibility of slipping; and we must therefore endeavour to obtain a solution, such that a certain surface of no flux which passes through the sharp edge shall also be a surface of constant pressure. This surface of no flux will either form the free boundary of the liquid, or will constitute a surface of separation between the moving liquid and a region of liquid at rest, and in the latter case will be a surface of discontinuity along which slipping must take place. The only problems of this class which have yet been solved are problems of two dimensional motion, and the method of solution is due to Kirchhoff[1] and depends on the properties of complex variables.

128. Any complex variable $x + \iota y$, may be represented geometrically by means of a vector drawn from the origin to the point whose rectangular coordinates are (x, y).

If we put $x = r\cos\theta$, $y = r\sin\theta$, the length of the vector will be r, and θ will be the angle which its direction makes with the axis of x. The quantities r and θ are respectively called the *modulus* and *amplitude* of the complex $x + \iota y$.

The sum of two vectors $x + \iota y$ and $a + \iota b$ is $x + a + \iota(y + b)$, which represents a vector drawn from the origin to the point $(x + a, y + b)$. Hence the sum of two vectors is represented by

[1] *Crelle*, vol. LXX.; and *Vorles. über Math. Phy.* Chapters XXI., XXII.

the diagonal of the parallelogram of which the two vectors are adjacent sides.

Similarly the difference between two vectors is represented by a line drawn from the origin, which is equal and parallel to the line joining the opposite extremities of the two vectors.

The product of the two vectors is

$$\begin{aligned}(x + \iota y)(a + \iota b) &= ax - by + \iota(bx + ay)\\ &= R(\cos\phi + \iota\sin\phi),\end{aligned}$$

where $\qquad R\cos\phi = ax - by, \quad R\sin\phi = bx + ay.$

Hence

$$R^2 = (a^2 + b^2)(x^2 + y^2),$$

$$\tan\phi = \frac{b/a + y/x}{1 - by/ax}.$$

Whence the product of two vectors is a vector whose length is equal to the square root of the product of the two vectors, and whose direction is inclined to the axis of x, at an angle which is equal to the sum of the inclinations of its factors.

Similarly the quotient of two vectors is a vector whose length is equal to the square root of the quotient of the two vectors, and whose direction is inclined to the axis of x, at an angle which is equal to the difference of the inclinations of the dividend and divisor.

129. Let z and w denote the two complexes $x + \iota y$ and $\phi + \iota\psi$; and let x and y be rectangular coordinates of a point P in a plane, which we shall call the plane of z; and let ϕ and ψ be rectangular coordinates of a point P' in another plane which we shall call the plane of w. Then if w and z be connected by any relation $w = f(z)$, it follows that if P trace out any curve in the plane of z, P' will trace out a corresponding curve in the plane of w.

130. Every function of a complex has a differential coefficient, for

$$\begin{aligned}f'(z) = \frac{dw}{dz} &= \frac{d\phi + \iota d\psi}{dx + \iota dy}\\ &= \frac{(d\phi/dx + \iota d\psi/dx)\,dx + (d\phi/dy + \iota d\psi/dy)\,dy}{dx + \iota dy}.\end{aligned}$$

And since

$$\frac{d\phi}{dy} + \iota\frac{d\psi}{dy} = \iota\left(\frac{d\phi}{dx} + \iota\frac{d\psi}{dx}\right),$$

this ratio is independent of the ratio dy/dx.

If ϕ and ψ be the velocity potential and current function of a liquid,

$$\frac{dw}{dz} = \frac{d\phi}{dx} + \iota \frac{d\psi}{dx} = u - \iota v.$$

Therefore $$\frac{dz}{dw} = \frac{1}{u - \iota v} = \frac{1}{q^2}(u + \iota v) = \zeta \text{ (say)},$$

where q is the resultant velocity of the liquid; hence the vector ζ represents the reciprocal of the velocity of the liquid.

131. In the class of problems which we are about to consider, the boundaries of the liquid consist partly of straight lines which constitute the fixed boundaries of the liquid, and along which the direction of the velocity is necessarily constant; and partly of the free surface of the liquid or of surfaces of discontinuity, which divide the moving liquid from the region of liquid at rest, and along which the pressure and consequently the magnitude of the velocity must be constant. Hence, if we choose the scale of measurement such that $q = 1$ along the latter surfaces, the boundaries will become transformed in the plane of ζ into an arc of a circle of unit radius, which corresponds to the free surface, or surfaces of discontinuity; and into the radii of this circle, which correspond to the fixed boundaries. The points where the radii meet the circle correspond to the points where the fixed and free boundaries intersect; also since the velocity must not become infinite, ζ can never vanish, and therefore the portion of the plane of ζ external to this circle and included between the two radii, corresponds to the portion of the plane of w occupied by the moving liquid.

Along the boundaries fixed and free, of the liquid in the plane of z, we must have $\psi = \alpha$, and $\psi = \beta$, where α and β are constants; hence the corresponding portion of the plane of w consists of the space included between the two parallel straight lines $\psi = \alpha$, $\psi = \beta$.

We must therefore endeavour to connect ζ and w by a relation, such that the above mentioned portions of the two planes of ζ and w shall correspond; and also that certain points in these two planes shall correspond to certain points in the plane of z. When this has been effected, the relation between z and w, which determines ϕ and ψ in terms of x and y, must be obtained by integration.

132. We shall define a lune as the space which is included between two circular arcs which meet but do not cross.

The angle of a lune is the angle at which the arcs meet.

Let $z = x + \iota y$, $z' = x' + \iota y'$, where (x, y), (x', y') are the rectangular coordinates of two points P, P' in the planes of z, z' respectively. We shall now show that if P trace out any lune of angle α in the plane of z, and P' trace out another lune of angle α' in the plane of z', it is possible to connect z and z' by a relation, such that the angular points of the two lunes shall correspond; and also that any third point on the perimeter of one lune shall correspond to any third point on the perimeter of the other.

The equation

$$Z' = \frac{AZ + B}{CZ + D} \text{ or } Z = \frac{-DZ' + B}{CZ' + A} \quad \text{..............(1)},$$

where A, B, C, D are complex constants, transforms any circle in the plane of Z into another circle in the plane of Z'. For if the point P describe a circle about the point $c = a + \iota b$ as centre, we must have

$$\text{mod}\,(Z - c) = \text{const.} \quad \text{.....................(2)}$$

or

$$(x - a)^2 + (y - b)^2 = \text{const.}$$

Substituting the value of Z in terms of Z' from (1), (2) becomes

$$\text{mod}\left(K\frac{Z' - C_1}{Z' - C_2}\right) = \text{const.} \quad \text{..................(3)},$$

where K, C_1, C_2 are new complex constants. Now if k, ρ_1, ρ_2 are the moduli of K, $Z' - C_1$, $Z' - C_2$, (3) may be written

$$\frac{k\rho_1}{\rho_2} = \text{const.},$$

whence P' moves so that the ratio of its distances from the two fixed points C_1, C_2 is constant, and therefore describes a circle.

Since (1) contains three disposable constants, viz. the ratios of the three quantities A, B, C, to D, it follows that these ratios may be chosen, so that a circle passing through three given points in the plane of Z shall correspond to a circle passing through three given points in the plane of Z'.

133. Let $$X + \iota Y = \mathfrak{Z} = \frac{z - c_1}{z - c_2} \quad \text{......................(4)}$$

where $$c_1 = a + \iota b, \quad c_2 = \alpha + \iota \beta.$$

Let A and B be the points c_1 and c_2. The vector $\mathfrak{Z}$ being the quotient of the two vectors AP and BP, is represented in the plane of $\mathfrak{Z}$ by a straight line whose inclination to the axis of X is equal to APB. Now if P describe a circle passing through A and B, the angle APB is constant, hence every circle passing through the points A and B in the plane of z, corresponds to a straight line passing through the origin in the plane of $\mathfrak{Z}$. Also if P and Q are any two points on two different circles passing through A and B, the inclination of the two corresponding lines in the plane of $\mathfrak{Z}$ is equal to $BQA - BPA$, that is to the angle of the lune $AQBPA$. Hence (4) transforms any lune in the plane of z into two straight lines in the plane of $\mathfrak{Z}$ whose inclination is equal to the angle of the lune.

If we put $$Z = \mathfrak{Z}^n,$$

the two straight lines in the plane of $\mathfrak{Z}$ become transformed into two straight lines in the plane of Z inclined at an angle n times as great; hence if α be the angle of the lune and $n = \pi/\alpha$, the equation

$$Z = \left(\frac{z - c_1}{z - c_2}\right)^{\frac{\pi}{\alpha}} \quad \text{.........................(5)}$$

transforms a lune in the plane of z whose angle is α and whose angular points are c_1, c_2 into a single straight line in the plane of Z.

Similarly if z' be any other plane, the equation

$$Z' = \left(\frac{z' - c_1'}{z' - c_2'}\right)^{\frac{\pi}{\alpha'}} \quad \text{.........................(6)}$$

transforms a lune in the plane of z' whose angle is α' and whose angular points are c_1', c_2' into a single straight line in the plane of Z'.

If therefore we substitute the values of Z, Z' from (5) and (6) in (1), the resulting equation transforms any lune of angle α in the plane of z into a lune of angle α' in the plane of z'; and by suitably choosing the ratios $A : B : C : D$, we may make any three points on the perimeter of one lune correspond to any three points on the perimeter of the other.

134. We must now notice some particular cases.

(i) Let $$z=\epsilon^w \text{ or } x+\iota y=\epsilon^{\phi+\iota\psi} \quad\ldots\ldots\ldots\ldots(7),$$

whence $$x=\epsilon^\phi \cos\psi,\quad y=\epsilon^\phi\sin\psi.$$

When $\psi=0$ or π, $y=0$; hence (7) transforms the two parallel straight lines $\psi=0$, $\psi=\pi$ in the plane of w into the single straight line $y=0$ in the plane of z.

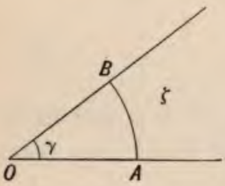
(ii) Let $$z=\sqrt{w} \text{ or } x+\iota y=\sqrt{\phi+\iota\psi} \quad\ldots\ldots\ldots\ldots(8).$$

Putting $$\phi=R\cos\chi,\quad \psi=R\sin\chi,$$

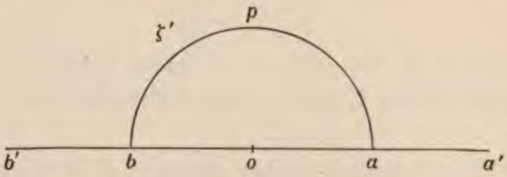
we obtain $$x=\sqrt{R}\cos\tfrac{1}{2}\chi,\quad y=\sqrt{R}\sin\tfrac{1}{2}\chi.$$

When $\sqrt{R}\sin\frac{1}{2}\chi=\text{const.}=c$, $y=c$; hence (8) transforms the confocal parabolas $\sqrt{R}\sin\frac{1}{2}\chi=c$ in the plane of w into the parallel straight lines $y=c$ in the plane of z. Now if $c=0$ the parabolas degenerate into a double line extending from the focus to ∞. Hence (8) transforms a straight line in the plane of w extending from a fixed point to infinity, into the whole of the axis of x, in the plane of z.

(iii) Let us now consider the portion of space bounded by the straight lines OA, OB in the plane of ζ, which is external to the circular arc AB.



If γ is the inclination of OA, OB, the equation $\zeta'=\zeta^{\pi/\gamma}$ transforms the two straight lines OA, OB in the plane of ζ into a single straight line in the plane of ζ'; and the arc AB into the semicircle ab. Hence the transformed region in the plane of ζ', is the portion of space lying



on the upper side of $a'b'$, and which is bounded by the semicircle and the infinite straight lines aa', bb'. This region may be regarded as a lune of angle $\frac{1}{2}\pi$, one of whose arcs is the semicircle apb; and whose other arc consists of the infinite lines aa', bb', which

may be regarded as an arc of a circle whose centre is at infinity. By (5), the equation

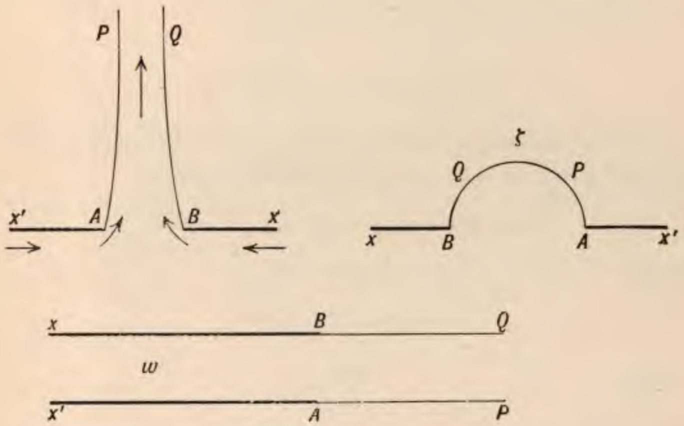
$$z = \left(\frac{\zeta' - 1}{\zeta' + 1}\right)^2$$

transforms this lune into a single straight line in the plane of z, hence the required transformation is

$$z = \left(\frac{\zeta^{\pi/\gamma} - 1}{\zeta^{\pi/\gamma} + 1}\right)^2 \quad \text{......................(9)}.$$

135. We shall now apply the preceding method to the solutions of some special problems.

A jet of liquid escapes by a slit AB from a large cistern of which the side is $x'x$; required the motion, which is supposed to be in two dimensions.



The figures show the corresponding lines in the planes of z, ζ and w; corresponding points being represented by the same letters in each of the three planes, and the fixed and free boundaries and their corresponding lines by thick and thin lines respectively. The lines $x'A$, Bx along which the direction of the velocity is invariable, are represented in the plane of ζ by the straight lines $x'A$, Bx; and the free surface of the jet along which the magnitude of the velocity is invariable and equal to unity, by the semicircle $APQB$. The portion of the plane of ζ lying above the line $xBQPAx'$, corresponds to the space occupied by the liquid. In the plane of w this space corresponds to the region contained

between the parallel straight lines $x'AP$ and xBQ. Let $\psi = 0$, $\psi = \pi$ be the stream lines $x'AP$, and xBQ: also let $\phi = 0$ be the equipotential surface passing through A and B.

In order to transform the region in the plane of ζ to that in the plane of w, we must put $\gamma = \pi$ in (9) and we obtain from (7) and (1),

$$\left(\frac{\zeta - 1}{\zeta + 1}\right)^2 = \frac{A\epsilon^w + B}{C\epsilon^w + D}.$$

Since a liquid flows from places of lower to places of higher velocity potential, the following conditions must be satisfied:

$$\text{(i)} \quad \phi = -\infty, \ \zeta = \infty, \quad \text{(ii)} \quad \phi = \infty, \ \zeta = -\iota,$$

$$\text{(iii)} \quad w = 0, \quad \zeta = 1, \quad \text{(iv)} \quad w = \iota\pi, \ \zeta = -1.$$

Of these (i) gives $B = D$; (ii) gives $A = -C$; and (iii) and (iv) both give $A = -B$; whence

$$\left(\frac{\zeta - 1}{\zeta + 1}\right)^2 = \frac{1 - \epsilon^w}{1 + \epsilon^w},$$

or
$$\zeta = \frac{dz}{dw} = \epsilon^{-w} + \sqrt{\epsilon^{-2w} - 1}.$$

Let θ be the angle which the tangent to AP makes with AB; along AP $q = 1$, $\psi = 0$, and ϕ is positive; hence

$$\cos\theta + \iota\sin\theta = \epsilon^{-\phi} + \iota\sqrt{1 - \epsilon^{-2\phi}},$$

whence
$$\cos\theta = \epsilon^{-\phi},$$
$$\sin\theta = \sqrt{1 - \epsilon^{-2\phi}}.$$

Also
$$\frac{d\phi}{ds} = 1,$$

therefore measuring s from A, we obtain

$$s = \phi$$

and
$$\frac{dx}{ds} = \cos\theta = \epsilon^{-s},$$

therefore
$$x = 1 - \epsilon^{-s} \quad \text{.........................(10)},$$

A being the origin. When $s = \infty$, $x = 1$; also since the final width of the jet is π, the width of the slit is $\pi + 2$.

The ratio of the final width of the jet to the width of the slit, is called the *coefficient of contraction* of the jet, which is therefore equal to $\pi/(\pi + 2)$ or $\cdot 611$.

Again $$\frac{dy}{ds} = \sin\theta = \sqrt{1-\epsilon^{-2s}},$$

$$y = \sqrt{1-\epsilon^{-2s}} - \tfrac{1}{2}\log\frac{1+\sqrt{1-\epsilon^{-2s}}}{1-\sqrt{1-\epsilon^{-2s}}} \quad\text{............ (11).}$$

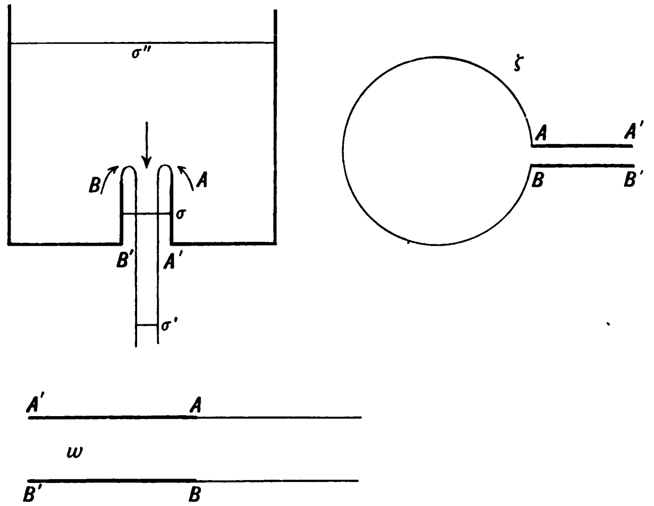
Eliminating s between (10) and (11), the equation of the free surface of the jet is

$$y = \sqrt{2x-x^2} - \tfrac{1}{2}\log\frac{1+\sqrt{2x-x^2}}{1-\sqrt{2x-x^2}}.$$

Also the radius of curvature is $\tan\theta$, which vanishes at the origin.

If we put $\zeta^{\pi/\alpha}$ for ζ we obtain the solution when the boundaries xB, $x'A$ are inclined at an angle α.

136. Let us now suppose that the conditions of the last example are varied by introducing a tube projecting inwards[1].



The containing vessel is supposed to be so large that we may disregard what takes place at the sides. The motion will then be as follows. The liquid will flow along the side $B'B$, and at B the direction of its velocity will begin to change, and the liquid will finally flow out in a stream whose section will be less than that of the tube.

[1] Helmholtz, *Phil. Mag.* Nov. 1868.

Since the aperture of the tube is supposed to be small, the curve in the plane of ζ which corresponds to the free boundaries may be approximately regarded as a circle, and if we put $\zeta' = \sqrt{\zeta}$ the space bounded internally by this circle and the lines AA', BB', will be transformed into the region in the plane of ζ in the last example. The solution in this case may be obtained from the last example by writing $\sqrt{\zeta}$ for ζ, and we obtain

$$\frac{dz}{dw} = \zeta = (\epsilon^{-w} + \sqrt{2\epsilon^{-2w} - 1})^2,$$

$$= 2\epsilon^{-2w} - 1 + 2\epsilon^{-w}\sqrt{\epsilon^{-2w} - 1}.$$

Along the free surface of the jet, we have

$$s = \phi$$

$$\cos\theta + \iota\sin\theta = 2\epsilon^{-2\phi} - 1 + 2\iota\epsilon^{-\phi}\sqrt{1 - \epsilon^{-2\phi}},$$

therefore
$$\frac{dx}{ds} = \cos\theta = 2\epsilon^{-2s} - 1,$$

$$x = 1 - s - \epsilon^{-2s},$$

$$\frac{dy}{ds} = \sin\theta = 2\epsilon^{-s}\sqrt{1 - \epsilon^{-2s}},$$

$$y = \epsilon^{-s}\sqrt{1 - \epsilon^{-2s}} + \sin^{-1}\epsilon^{-s} + y',$$

the middle point of AB being the origin. When $s = \infty$, $y = y'$, so that $2y'$ is the final breadth of the stream and is therefore equal to π; when $s = 0$, $y = \frac{1}{2}\pi + y' = \pi$, whence $AB = 2\pi$, and the coefficient of contraction $= \frac{1}{2}$.

137. Lord Rayleigh[1] has shown that if the vessel were of finite dimensions, the coefficient of contraction must always be greater than $\frac{1}{2}$; for let σ'' be the area of a section of the vessel so far removed from the orifice that the velocity over it is sensibly constant and equal to v''. Let v', σ' be the ultimate velocity and section of the jet, σ the section of the tube. The equation of continuity gives

$$v'\sigma' = v''\sigma''.$$

By the principle of energy

$$p = \tfrac{1}{2}(v'^2 - v''^2),$$

and by the principle of momentum

$$p\sigma = \sigma' v'^2 - \sigma'' v''^2.$$

[1] "The Contracted Vein," *Phil. Mag.* Dec. 1876.

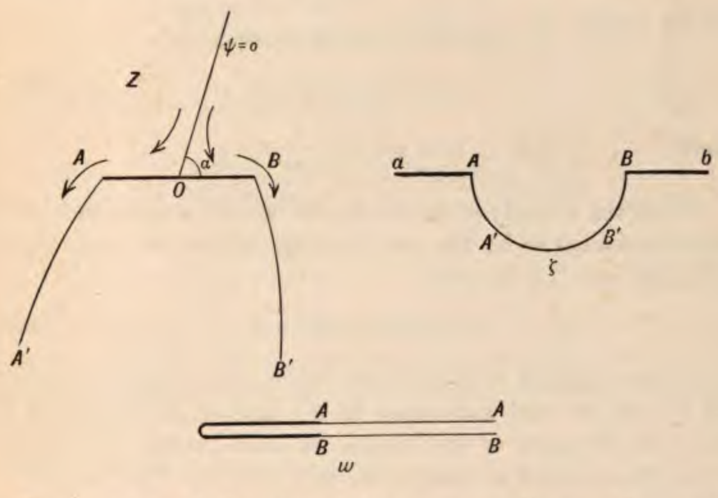
From these equations we obtain

$$\frac{2}{\sigma} = \frac{1}{\sigma'} + \frac{1}{\sigma''},$$

which shows that the section of the tube is an harmonic mean between the sections of the cylinder and jet. When $\sigma'' = \infty$, $\sigma'/\sigma = \frac{1}{2}$ as before.

138. When a rectangular lamina is held fixed in a stream which meets it obliquely, there will be a region of dead water behind the lamina, which will be at rest, and the total pressure on the lamina will be due to the difference of pressures upon its anterior and posterior faces.

The stream line $\psi = 0$ meets the lamina at some point O and then divides, each branch following the lamina to its edges, and afterwards forming the boundary between the moving liquid and the dead water behind the lamina.



The portion of the plane of ζ corresponding to the moving liquid is that which lies below the semicircle $AA'B'B$ and the two infinite lines Bb, Aa; and the points $\pm\infty$ correspond to O. The whole of the plane of w corresponds to the portion occupied by the moving liquid, with the exception of the double line shown in the figure, which may be regarded as the limiting form of a parabola.

Let α be the angle at which the stream meets the lamina; since the equation $w' = \sqrt{w}$ converts the double line in the plane

of w, into a single straight line in the plane of w', we must put

$$\left(\frac{\zeta-1}{\zeta+1}\right)^2=\frac{A\sqrt{w}+B}{C\sqrt{w}+D}.$$

The conditions to be satisfied are

$$\text{(i)}\quad \phi=\pm\infty,\quad \zeta=\cos\alpha-\iota\sin\alpha,$$

$$\text{(ii)}\quad w=0,\qquad \zeta=\infty.$$

From (i) we obtain

$$A=-C\tan^2\tfrac{1}{2}\alpha,$$

and from (ii) $B=D$, whence

$$\left(\frac{\zeta-1}{\zeta+1}\right)^2=\frac{-C\sqrt{w}\tan^2\frac{1}{2}\alpha+B}{C\sqrt{w}+B}.$$

Let $$\frac{C}{B}=\sqrt{K}\,(1+\cos\alpha),\ \omega=Kw,$$

and we obtain $$\left(\frac{\zeta-1}{\zeta+1}\right)^2=\frac{1-(1-\cos\alpha)\sqrt{\omega}}{1+(1+\cos\alpha)\sqrt{\omega}},$$

or $$\zeta=\Omega+\sqrt{\Omega^2-1}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(12),$$

where $$\Omega=\cos\alpha+\frac{1}{\sqrt{\omega}}.$$

When the velocity of the stream at infinity is equal to V, which will be supposed to be the case in what follows, we must change ζ into ζV, and (12) becomes

$$\zeta V=\Omega+\sqrt{\Omega^2-1}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(13).$$

In the plane of z let O be the origin, OB the axis of x; along AB ζ must be real and equal to u^{-1}, and at A and B $\zeta=V^{-1}$. Hence at all points of the lamina we must have $\Omega>1$, and at A and B, $\Omega=-1$ and $+1$ respectively.

Let l be the breadth of the lamina, then since along AB $K\phi=\omega$ and $d\phi/dx=u$,

$$l=\int\frac{d\phi}{u}=\int(\Omega+\sqrt{\Omega^2-1})\,\frac{d\omega}{VK}\ \ldots\ldots\ldots\ldots(14),$$

the limits of integration being determined by

$$\Omega=\cos\alpha+\frac{1}{\sqrt{\omega}}=\pm 1.$$

If β be a new variable such that

$$\beta = \sqrt{\omega}\sin^2\alpha - \cos\alpha,$$

the limits of β will be ± 1, and we obtain

$$\tfrac{1}{2}VKl = \int_{-1}^{1}\{(\beta+\cos\alpha)\cos\alpha + \sin^2\alpha + \sqrt{1-\beta^2}\sin\alpha\}\operatorname{cosec}^4\alpha\, d\beta.$$

Whence $$K = \frac{4+\pi\sin\alpha}{Vl\sin^4\alpha}.$$

Along the lines AA', BB' the pressure $p' = \rho(C - \tfrac{1}{2}V^2)$, which must be equal to the hydrostatic pressure of the dead water. At the surface of the lamina,

$$\frac{p}{\rho} = C - \tfrac{1}{2}u^2$$

$$= \frac{p'}{\rho} + \tfrac{1}{2}(V^2 - u^2).$$

Hence the total pressure on the lamina is,

$$\varpi = \int(p - p')\,dx = \tfrac{1}{2}\rho\int(V^2 - u^2)\frac{d\phi}{u}$$

$$= \rho V\int(\Omega^2 - 1)^{\frac{1}{2}}\,d\phi$$

$$= \frac{2V\rho}{K\sin^3\alpha}\int_{-1}^{1}\sqrt{1-\beta^2}\,d\beta = \frac{\pi V\rho}{K\sin^3\alpha}$$

$$= \frac{\pi V^2 l\rho\sin\alpha}{4+\pi\sin\alpha} \quad\text{..............................(15)},$$

which determines the resistance which the lamina offers to the stream, and shows that it depends partly upon the square of the velocity and partly upon the angle which the stream makes with the lamina.

The moment of the pressure is

$$G = \frac{2V\rho}{K\sin^3\alpha}\int_{-1}^{1} x\sqrt{1-\beta^2}\,d\beta.$$

Now by (14),

$$\tfrac{1}{2}VKx = \int\{\cos\alpha\,(\beta+\cos\alpha) + \sin^2\alpha + \sqrt{1-\beta^2}\sin\alpha\}\operatorname{cosec}^4\alpha\,d\beta.$$

Hence, if the origin be suitably chosen, the value of x will be

$$x = \frac{\beta^2\cos\alpha + 2\beta + \{\beta\sqrt{1-\beta^2} + \sin^{-1}\beta\}\sin\alpha}{VK\sin^4\alpha}.$$

The odd terms in β contribute nothing to the integral, and therefore

$$G=\frac{2\rho}{K^2\sin^7\alpha}\int_{-1}^{1}\beta^2\sqrt{1-\beta^2}\cos\alpha d\beta$$

$$=\frac{\pi\rho\cos\alpha}{4K^2\sin^7\alpha}=\frac{\varpi\cos\alpha}{4KV\sin^4\alpha}.$$

The distance of the middle point of the lamina from the origin is $\cos\alpha/VK\sin^4\alpha$; hence the distance of the centre of pressure from line middle point is

$$-\frac{3\cos\alpha}{4KV\sin^4\alpha}=-\frac{3l\cos\alpha}{4(4+\pi\sin\alpha)}.$$

If $\frac{1}{2}\pi>\alpha>0$, the negative sign shows that the centre of pressure is on the upstream side of the middle point; hence if the lamina be free to turn about an axis parallel to its edges whose distance from the middle point is

$$x=\frac{3l\cos\alpha}{4(4+\pi\sin\alpha)},\quad\text{.....................(16)},$$

it will be in equilibrium. If $\alpha=\frac{1}{2}\pi$, $x=0$; and the lamina will set itself transversely to the stream. When $\alpha=0$, x is a maximum and is equal to $3l/16$, in which case the axis divides the lamina in the ratio 11 : 5.

139. The results of equations (15 and 16), which are due to Lord Rayleigh[1], may be stated in another form as follows. "If the axis of suspension divide the width in a more extreme ratio than 11 : 5, there is but one position of stable equilibrium, that namely in which the lamina is parallel to the stream with the narrower portion directed upwards. If the axis be situated exactly at the point which divides the width in the ratio 11 : 5, this position becomes neutral, in the sense that for small displacements the force of restitution is of the second order, but the equilibrium is in reality stable. When the axis is still nearer the centre of figure, the position parallel to the stream becomes unstable, and is replaced by two inclined positions making with the stream equal angles, which increase from zero to a right angle as the axis moves towards the centre. With the centre line itself for axis, the lamina can only remain at rest when transverse to the stream although of course with either face turned upwards[2]."

[1] "On the resistance of fluids," *Phil. Mag.* Dec. 1876. [2] *Ibid.*

140. In order to obtain the intrinsic equation of the surface of separation, we have along this surface

$$\zeta V = \Omega + \iota\sqrt{1-\Omega^2}.$$

Therefore $$\frac{u}{V} = \Omega = \cos\alpha + \frac{1}{\sqrt{K\phi}}.$$

Now $$\frac{d\phi}{ds} = V,$$

therefore $$\phi = V(s+c),$$

and therefore $$\frac{dx}{ds} = \cos\theta = \cos\alpha \pm \frac{1}{\sqrt{\{VK(s+c)\}}}.$$

The constant c is to be determined from the fact that when $s = 0$, $\cos\theta = \pm 1$. In the case of perpendicular incidence, we have $c = 1/VK$, whence

$$\frac{dx}{ds} = \sqrt{\frac{c}{s+c}},$$

or $$x = 2(cs+c^2)^{\frac{1}{2}} + \text{constant},$$

from which it appears that x does not approach a finite limit as s increases indefinitely.

The methods of this chapter only apply when the motion is in two dimensions; so far as I am aware, no problem of this class has been solved when the motion is in three dimensions.

MISCELLANEOUS EXAMPLES.

1. If u, v, w, ϕ are any functions of x, y, z, prove that $udx + vdy + wdz - d\phi$ has an integrating factor; hence show that if u, v, w be the velocities of a fluid, then along any vortex line

$$udx + vdy + wdz = d\phi.$$

2. If in an infinite mass of homogeneous incompressible fluid in equilibrium under finite fluid pressure only, an indefinitely long cylindrical column be suddenly annihilated, prove that no motion will take place.

3. Prove that the velocity potential due to a unit source placed outside a sphere of radius a, and at a distance f from its centre is

$$\phi = -(r^2 - 2fr\cos\theta + f^2)^{-\frac{1}{2}} - af^{-1}(r^2 - 2cr\cos\theta + c^2)^{-\frac{1}{2}}$$
$$+ a^{-1}\{\log[c - r\cos\theta + (r^2 - 2cr\cos\theta + c^2)^{\frac{1}{2}}] - \log r(1-\cos\theta)\},$$

where (r, θ) are polar coordinates referred to the centre of the sphere as origin, and $c = a^2/f$.

4. Prove that the rate at which the energy of a mass of liquid, contained within an imaginary closed surface described in the liquid is increasing, is equal to

$$\iint (p + \rho V)\, q \cos \epsilon dS,$$

where p is the pressure, V the potential of the impressed forces, q the resultant velocity at any point of S, and ϵ is the angle between the direction of q and the normal to S drawn outwards.

5. If a, b, c be curvilinear coordinates of any point (x, y, z) of a liquid, such that the lines of flow are the intersections of the surfaces $b = \text{const.}$, $c = \text{const.}$; apply § 39 to prove that when the motion of the liquid is *not* steady, a first integral of the general equations of motion is

$$\frac{p}{\rho} + V + \tfrac{1}{2}q^2 + \tfrac{1}{2}\int \frac{q}{J}\frac{dq}{dt}\, da = F(b, c, t),$$

where
$$J = \frac{d(a, b, c)}{d(x, y, z)}.$$

6. If the molecular rotation of a mass of liquid which completely fills a rigid circular cylinder be equal to $\tfrac{1}{2}r^{-1}F'(r)$, where $r^{-1}F'(r)$ is any function of r which does not become infinite within the cylinder; prove that the paths of individual particles of liquid are circles described in periodic time

$$2\pi r^2/F(r).$$

7. In § 135, if v be the velocity at any point on the middle line of the jet, whose distance from the orifice is y, prove that

$$y = \frac{1}{v} - \log\frac{1+v}{1-v},$$

the ultimate velocity of the jet being unity, and the scale of measurement being such that $\pi + 2$ is the width of the orifice.

CHAPTER VII.

ON THE KINEMATICS OF SOLID BODIES MOVING IN A LIQUID.

141. In the present chapter we shall obtain expressions for the velocity potential, in a variety of cases in which a liquid is bounded externally or internally by moving solids, when the motion is in three dimensions. We shall suppose that the motion of the liquid is irrotational and acyclic, and consequently the motion will be completely determined by means of a velocity potential ϕ which must satisfy the following conditions;

(i) ϕ must be a single valued function, which at all points of the liquid satisfies the equation $\nabla^2\phi = 0$;

(ii) ϕ and its first derivatives must be finite and continuous at all points of the liquid, and must vanish at infinity if any portion of the liquid extends to infinity;

(iii) At all points of the liquid which are in contact with a moving solid, $d\phi/dn$ must be equal to the normal velocity of the solid, where dn is an element of the normal to the solid drawn outwards; if any portion of the liquid is in contact with fixed boundaries, $d\phi/dn$ must be zero at every point of these fixed boundaries.

142. Let us now suppose that a single solid is in motion in an infinite liquid.

Let Ox, Oy, Oz be three rectangular axes *fixed* in the solid, and let ϕ_1 be the velocity potential when the solid is moving with unit velocity parallel to Ox, and let χ_1 be the velocity potential when the solid is rotating with unit angular velocity about Ox. Let ϕ_2, ϕ_3, χ_2, χ_3 be similar quantities with respect to Oy and Oz. Also let u, v, w be the linear velocities of the solid parallel to, and ω_1, ω_2, ω_3 be its angular velocities about the axes.

The velocity potential of the whole motion will be

$$\phi = u\phi_1 + v\phi_2 + w\phi_3 + \omega_1\chi_1 + \omega_2\chi_2 + \omega_3\chi_3 \quad \text{.........(1)}.$$

For if λ, μ, ν be the direction cosines of the normal at any point x, y, z on the surface of the solid, we must have at the surface

$$\frac{d\phi_1}{dn} = \lambda, \quad \frac{d\phi_2}{dn} = \mu, \quad \frac{d\phi_3}{dn} = \nu,$$

$$\frac{d\chi_1}{dn} = \nu y - \mu z, \quad \frac{d\chi_2}{dn} = \lambda z - \nu x, \quad \frac{d\chi_3}{dn} = \mu x - \lambda y.$$

Hence $\dfrac{d\phi}{dn} = (u - y\omega_3 + z\omega_2)\lambda + (v - z\omega_1 + x\omega_3)\mu + (w - x\omega_2 + y\omega_1)\nu$

$= \text{normal velocity of the solid.}$

143. *To find the velocity potential when a sphere of radius a is moving parallel to the axis of x*[1].

Let u be the velocity of the sphere, a its radius, θ the angle which the radius to any point on its surface makes with Ox, then at the surface,

$$\frac{d\phi}{dn} = u \cos\theta,$$

or

$$\frac{d\phi}{dr} = u \cos\theta \quad \text{......................(2)},$$

when $r = a$.

Since the motion is symmetrical with respect to Ox, and the velocity must vanish at infinity, ϕ must be of the form

$$\phi = \frac{A_0}{r} + \frac{A_1 P_1}{r^2} + \frac{A_2 P_2}{r^3} + \ldots\ldots$$

where P_n is the zonal harmonic of degree n. Substituting in (2), we obtain

$$-\frac{A_0}{a^2} - \frac{2A_1 \cos\theta}{a^3} - \&c. = u \cos\theta,$$

whence $A_0 = A_2 = \&c. = 0,$

and $A_1 = -\tfrac{1}{2}ua^3,$

therefore

$$\phi = -\tfrac{1}{2}ua^3 \frac{\cos\theta}{r^2}$$

$$= -\tfrac{1}{2}ua^3 \frac{x}{r^3} \quad \text{..........................(3)}.$$

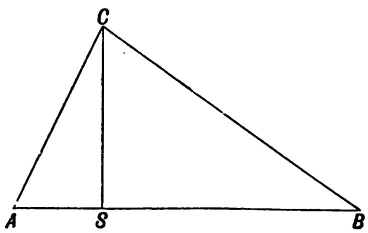
[1] Poisson, "Mémoire sur les mouvements simultanés d'un pendule et de l'air environnant," *Mém. de l'Acad. des Sciences*, Paris, vol. IX. p. 521.

144. If the sphere were moving with component velocities u, v, w, parallel to the axes, the velocity potential would be

$$\phi = -\frac{a^3}{2r^3}(ux + vy + wz).$$

This expression is the velocity potential of a doublet situated at the centre of the sphere, whose axis coincides with and whose source end is turned towards the direction of motion of the sphere.

145. The velocity potential may be determined by the method of images, when the solid, which is formed by the revolution about the line joining their centres, of two spheres which intersect at right angles, is moving parallel to its axis[1].



Let A and B be the centres of the two spheres, C a point on their circle of intersection; then if CS is perpendicular to AB, S is the common image of B and A with respect to the spheres A and B. Let $AC = a$, $BC = b$,

$$AB = c = \sqrt{a^2 + b^2},$$

and let u be the velocity of the solid along AB; also let (r, θ), (r_1, θ_1), (r_2, θ_2) be the polar coordinates of any point P referred to B, S and A respectively as origin.

The velocity potential due to the motion of B alone is

$$\phi_1 = -\frac{ub^3}{2r^2}\cos\theta,$$

which is the same as that due to a doublet of strength $\frac{1}{2}ub^3$ at B.

The image of this in A is a doublet at S of strength

$$-\tfrac{1}{2}ub^3\left(\frac{a}{AB}\right)^3,$$

and the image of this in B is a doublet at A of strength

$$\tfrac{1}{2}ub^3\left(\frac{ab}{AB\,.\,BS}\right)^3 = \tfrac{1}{2}ua^3.$$

This is precisely what is required to give the requisite normal velocity over A and B, whence

$$\phi = -\tfrac{1}{2}u\left(\frac{b^3\cos\theta}{r^2} - \frac{a^3b^3\cos\theta_1}{c^3r_1^2} + \frac{a^3\cos\theta_2}{r_2^2}\right).$$

[1] Stokes, *Math. and Phys. Papers*, vol. I. p. 230.

146. The motion of two spheres will be discussed in Chapter XI., but when the space between two concentric spheres is filled with liquid, and the spheres are moved in any manner, the velocity potential of the *initial* motion can be obtained as follows[1].

Let a and b be the radii of the outer and inner spheres respectively, O their common centre; and let the outer sphere be moved with velocity u along any direction OA, also let the inner sphere be moved with velocity v along a direction OB which is perpendicular to OA. Let θ be the angle which the radius to any point P makes with OA, χ the angle which the plane OAP makes with the plane OAB.

The surface conditions are

$$\left(\frac{d\phi}{dr}\right)_a = u\cos\theta, \quad \left(\frac{d\phi}{dr}\right)_b = v\sin\theta\cos\chi \quad\dots\dots\dots\dots(4).$$

The function

$$\phi = \left(Ar + \frac{B}{r^2}\right)\cos\theta + \left(Cr + \frac{D}{r^2}\right)\sin\theta\cos\chi$$

satisfies Laplace's equation. Substituting in the first of (4) we must have

$$A - \frac{2B}{a^3} = u, \quad C - \frac{2D}{a^3} = 0,$$

and from the second of (4)

$$A - \frac{2B}{b^3} = 0, \quad C - \frac{2D}{b^3} = v,$$

whence $\quad A = ua^3/(a^3 - b^3), \quad B = \tfrac{1}{2}ua^3b^3/(a^3 - b^3),$

$$C = -vb^3/(a^3 - b^3), \quad D = -\tfrac{1}{2}va^3b^3/(a^3 - b^3)$$

and $\quad \phi = \dfrac{ua^3}{a^3 - b^3}\left(r + \dfrac{b^3}{2r^2}\right)\cos\theta - \dfrac{vb^3}{a^3 - b^3}\left(r + \dfrac{a^3}{2r^2}\right)\sin\theta\cos\chi.$

147. The velocity potential due to the motion of an ellipsoid in an infinite liquid was first obtained by Green in 1833, for the case of translation only[2]; the solution was completed for the case of rotation by Clebsch in 1856[3].

(i) Let the ellipsoid move parallel to the axis of x with unit velocity.

[1] Stokes, "On some cases of fluid motion," *Trans. Camb. Phil. Soc.*, VIII. p. 105.

[2] "Researches on the vibration of pendulums in fluid media," *Trans. Roy. Soc. Edin.*, 1833.

[3] "Ueber die Bewegung einer Ellipsoids in einer tropfbaren Flüssigkeit," *Crelle*, LII. p. 103.

If V be the potential at an external point of a homogeneous ellipsoid of attracting matter of unit density, the equation of whose bounding surface is

$$(x/a)^2 + (y/b)^2 + (z/c)^2 = 1,$$

$$V = \pi abc\int_\lambda^\infty \left(\frac{x^2}{a^2+\psi} + \frac{y^2}{b^2+\psi} + \frac{z^2}{c^2+\psi} - 1\right)\frac{d\psi}{\{(a^2+\psi)(b^2+\psi)(c^2+\psi)\}^{\frac{1}{2}}},$$

where λ is the positive root of the equation

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1 \quad\text{..............}\quad (5).$$

The potential at an internal point is obtained by putting $\lambda = 0$ in the definite integral. We shall write this expression in the form

$$V = \tfrac{1}{2}(A_\lambda x^2 + B_\lambda y^2 + C_\lambda z^2) - H_\lambda \quad\text{..............}(6),$$

where

$$\left.\begin{aligned} A_\lambda &= 2\pi abc\int_\lambda^\infty \frac{d\psi}{(a^2+\psi)P}\ \&c. \\ H_\lambda &= \pi abc\int_\lambda^\infty \frac{d\psi}{P} \end{aligned}\right\} \quad\text{............}(7),$$

$$P = \{(a^2+\psi)(b^2+\psi)(c^2+\psi)\}^{\frac{1}{2}},$$

and we shall drop the suffix λ, when these quantities refer to an internal point.

If p is the perpendicular from the centre on to the tangent plane at x, y, z; the surface condition is,

$$\frac{d\phi_1}{dn} = l = \frac{px}{a^2},$$

or

$$\frac{x}{a^2}\frac{d\phi_1}{dx} + \frac{y}{b^2}\frac{d\phi_1}{dy} + \frac{z}{c^2}\frac{d\phi_1}{dz} = \frac{x}{a^2} \quad\text{.................}(8).$$

Since $A_\lambda x$ is the x-component of the attraction of the ellipsoid, this quantity obviously satisfies conditions (i) and (ii) of § 142; we may therefore assume that

$$\phi_1 = \alpha A_\lambda x.$$

Hence at the surface

$$\frac{d\phi_1}{dx} = \alpha\left(A - \frac{2\pi x}{a^2}\frac{d\lambda}{dx}\right),$$

$$\frac{d\phi_1}{dy} = -\frac{2\pi\alpha x}{a^2}\frac{d\lambda}{dy},$$

$$\frac{d\phi_1}{dz} = -\frac{2\pi\alpha x}{a^2}\frac{d\lambda}{dz}.$$

Differentiating (5) with respect to x, and then putting $\lambda = 0$, we obtain

$$\frac{d\lambda}{dx} = \frac{2xp^2}{a^2}, \quad \frac{d\lambda}{dy} = \frac{2yp^2}{b^2}, \quad \frac{d\lambda}{dz} = \frac{2zp^2}{c^2};$$

hence the left-hand side of (8) becomes

$$\alpha x (A - 4\pi)/a^2,$$

whence
$$\alpha = (A - 4\pi)^{-1},$$

and
$$\phi_1 = \frac{A_\lambda x}{A - 4\pi}.$$

It therefore follows that if the ellipsoid is moving with velocities, u, v, w parallel to the axes

$$\phi = \frac{A_\lambda ux}{A - 4\pi} + \frac{B_\lambda vy}{B - 4\pi} + \frac{C_\lambda wz}{C - 4\pi} \quad \text{...............(9)}.$$

(ii) Let the ellipsoid be rotating with unit angular velocity about Ox; then the surface condition is

$$\frac{d\chi_1}{dn} = \omega_1 (ny - mx) = \frac{\omega_1 pyz (b^2 - c^2)}{b^2c^2} \quad \text{...........(10)}.$$

Writing for a moment Y and Z for $B_\lambda y$ and $C_\lambda z$, it can easily be shown that the function $zY - yZ$ satisfies Laplace's equation,

for
$$\nabla^2 (zY - yZ) = 2\left(\frac{dY}{dz} - \frac{dZ}{dy}\right)$$
$$= 2\left(\frac{d^2V}{dzdy} - \frac{d^2V}{dydz}\right) = 0,$$

also at great distance from the origin Y and Z are at least of the order r^{-2}, and therefore χ_1 is at least of the order r^{-1} and therefore vanishes at infinity.

Let us therefore assume

$$\chi_1 = \alpha' (zY - yZ) = \alpha' yz (B_\lambda - C_\lambda),$$

then at the surface

$$\frac{d\chi_1}{dn} = \alpha' \left(Y\frac{dz}{dn} - Z\frac{dy}{dn} + z\frac{dY}{dn} - y\frac{dZ}{dn}\right)$$
$$= \frac{p\alpha' yz}{b^2c^2}\{(B - C)(b^2 + c^2) + 4\pi (b^2 - c^2)\}.$$

Substituting in (10) we obtain

$$\alpha' = \frac{(b^2 - c^2)}{(B - C)(b^2 + c^2) + 4\pi (b^2 - c^2)},$$

therefore
$$\chi_1 = \frac{(b^2 - c^2)(B_\lambda - C_\lambda) yz}{(B - C)(b^2 + c^2) + 4\pi (b^2 - c^2)} \quad \text{.........(11)}.$$

The functions χ_2, χ_3 can be written down from symmetry.

148. The quantities A_λ, B_λ and C_λ may be expressed in terms of elliptic functions of the first and second kinds; but the most important case is when the ellipsoid is one of revolution.

(i) If we put $b = c < a$, the surface becomes an ovary ellipsoid and

$$A_\lambda = 2\pi ac^2 \int_\lambda^\infty \frac{d\psi}{(a^2+\psi)^{\frac{3}{2}}(c^2+\psi)}$$

$$= \frac{4\pi ac^2}{(a^2-c^2)^{\frac{3}{2}}} \int_\nu^\infty \frac{d\nu}{\nu^2(\nu^2-1)},$$

if $(a^2+\lambda)^{\frac{1}{2}} = (a^2-c^2)^{\frac{1}{2}}\nu$; therefore

$$A_\lambda = \frac{4\pi(1-e^2)}{e^3}\left(\tfrac{1}{2}\log\frac{\nu+1}{\nu-1} - \frac{1}{\nu}\right) \qquad\text{............(12)},$$

where e is the excentricity of the generating ellipse. Also

$$B_\lambda = C_\lambda = 2\pi ac^2 \int_\lambda^\infty \frac{d\psi}{(a^2+\psi)^{\frac{1}{2}}(c^2+\psi)^2}$$

$$= \frac{4\pi(1-e^2)}{e^3}\int_\nu^\infty \frac{d\nu}{(\nu^2-1)^2}$$

$$= \frac{2\pi(1-e^2)}{e^3}\left(\frac{\nu}{\nu^2-1} - \tfrac{1}{2}\log\frac{\nu+1}{\nu-1}\right) \qquad\text{......(13)}.$$

(ii) If we put $a = b > c$, so that the surface becomes a planetary ellipsoid we obtain

$$A_\lambda = B_\lambda = 2\pi a^2 c \int_\lambda^\infty \frac{d\psi}{(a^2+\psi)^2(c^2+\psi)^{\frac{1}{2}}}$$

$$= \frac{4\pi a^2 c}{(a^2-c^2)^{\frac{3}{2}}}\int_\nu^\infty \frac{d\nu}{(1+\nu^2)^2},$$

if $(c^2+\lambda)^{\frac{1}{2}} = (a^2-c^2)^{\frac{1}{2}}\nu$; therefore

$$A_\lambda = \frac{2\pi(1-e^2)^{\frac{1}{2}}}{e^3}\left(\cot^{-1}\nu - \frac{\nu}{\nu^2+1}\right) \qquad\text{............(14)},$$

$$C_\lambda = \frac{4\pi(1-e^2)^{\frac{1}{2}}}{e^3}\left(\frac{1}{\nu} - \cot^{-1}\nu\right) \qquad\text{..................(15)}.$$

It will be observed that in the case of an ovary ellipsoid $\nu = e'^{-1}$, where e' is the excentricity of the generating ellipse of the confocal ellipsoid which passes through the point (x, y, z); and that in the case of a planetary ellipsoid

$$\nu = \sqrt{1-e'^2}/e'.$$

149. If $c=0$ the planetary ellipsoid becomes a disc, and $\phi_1 = 0$; hence a disc which moves parallel to itself cuts through the liquid without producing any motion.

To find the velocity potential when the disc is moving perpendicularly to its plane, we observe that at the surface $\nu = 0$; hence when c and ν are small $c = a\nu$, therefore

$$C - 4\pi = \frac{4\pi c}{a}\left(\frac{a}{c} - \tfrac{1}{2}\pi\right) - 4\pi = -\frac{2\pi^2 c}{a},$$

therefore $$\phi = -\frac{2w}{\pi} z\left(\frac{1}{\nu} - \cot^{-1}\nu\right).$$

If μ, ν are elliptic coordinates, this equation may be written[1]

$$\phi = -\frac{2wa}{\pi}(1 - \nu\cot^{-1}\nu)\mu \quad\text{..............(16)},$$

$$= -\frac{2wa}{\pi} q_1(\nu) P_1(\mu).$$

By § 99 (14) and § 110 (31), the velocity perpendicular to the hyperboloid $\mu = \text{const.}$ is

$$\frac{1}{a}\sqrt{\frac{1-\mu^2}{\nu^2+\mu^2}}\frac{d\phi}{d\mu} = -\frac{2w}{\pi}\sqrt{\frac{1-\mu^2}{\nu^2+\mu^2}}(1 - \nu\cot^{-1}\nu).$$

At all points in the plane $z=0$ which do not lie on the disc, $\mu = 0$, and the velocity perpendicular to this plane

$$= -\frac{2w}{\pi\nu}(1 - \nu\cot^{-1}\nu),$$

which becomes infinite when $\nu = 0$. The velocity is therefore infinite at the edges, as we should expect since the liquid is supposed to move according to the electrical law of flow.

The solution for a stream flowing past a fixed disc behind which there is a region of dead water, has not yet been discovered.

[1] The function $q_n(\nu)$ is a spheroidal harmonic of the second kind, and is equal to $(-1)^{\frac{1}{2}(n+1)} Q_n(\iota\nu)$ where $Q_n(\nu)$ is a zonal harmonic of the second kind. The potential at an external point of any distribution of electricity upon an oblate spheroid which is symmetrical with respect to the axis of the spheroid, can be expanded in a series of terms of the type $q_n(\nu) P_n(\mu)$.

150. *To find the velocity potential when liquid is contained in an ellipsoidal cavity which is rotating about its centre.*

Here $$\frac{d\chi_1}{dn} = pyz\left(\frac{1}{c^2} - \frac{1}{b^2}\right).$$

Assume $$\chi_1 = Ayz.$$

Then $$\frac{d\chi_1}{dn} = \frac{py}{b^2}\frac{d\chi_1}{dy} + \frac{pz}{c^2}\frac{d\chi_1}{dz}$$

$$= Apyz\left(\frac{1}{b^2} + \frac{1}{c^2}\right).$$

Equating these two values of $d\chi_1/dn$, we obtain

$$A = \frac{b^2 - c^2}{b^2 + c^2}.$$

Hence $$\chi_1 = \frac{b^2 - c^2}{b^2 + c^2}yz.$$

This value of χ_1 satisfies Laplace's equation, and is such that the velocities are finite and continuous at all points of the liquid. Hence

$$\phi = \omega_1\frac{b^2 - c^2}{b^2 + c^2}yz + \omega_2\frac{c^2 - a^2}{c^2 + a^2}zx + \omega_3\frac{a^2 - b^2}{a^2 + b^2}xy \text{ ......(17)}.$$

151. Let us now suppose that the space between two concentric coaxial and confocal ellipsoids is filled with liquid, and that the inner and outer ellipsoids are suddenly moved with velocities U and V respectively parallel to the axis of z[1].

Let the accented and unaccented letters refer to the outer and inner ellipsoids respectively; and let

$$\phi = Mz + NC_\lambda z.$$

The surface conditions are

$$\frac{d\phi}{dn} = U\frac{pz}{c^2}, \quad \frac{d\phi}{dn} = V\frac{p'z'}{c'^2}.$$

From the first equation we obtain

$$M + N(C - 4\pi) = U,$$

and from the second

$$M + N(C' - 4\pi) = V,$$

whence $$M = \frac{U(C' - 4\pi) - V(C - 4\pi)}{C' - C},$$

$$N = -\frac{U - V}{C' - C},$$

[1] Greenhill, "Fluid motion between confocal elliptic cylinders and confocal ellipsoids," *Quart. Journ.* vol. XVI. p. 227.

and therefore

$$\phi = \frac{U(C' - 4\pi) - V(C - 4\pi) - (U - V)C_\lambda}{C' - C} z \text{ ......(18).}$$

If the outer ellipsoid were rotating about the axis of z with angular velocity Ω, and the inner with angular velocity ω, the surface conditions would be

$$\frac{d\chi}{dn} = \omega\, pxy \left(\frac{1}{b^2} - \frac{1}{a^2}\right), \quad \frac{d\chi}{dn} = \Omega\, p'xy \left(\frac{1}{b'^2} - \frac{1}{a'^2}\right).$$

We must therefore assume

$$\phi = Mxy + N(B_\lambda - A_\lambda)\, xy.$$

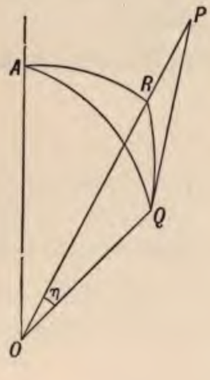
From the first equation we obtain

$$\left.\begin{aligned} &\{M + N(B - A)\}\left(\frac{1}{a^2} + \frac{1}{b^2}\right) - 4\pi N\left(\frac{1}{b^2} - \frac{1}{a^2}\right) = \omega\left(\frac{1}{b^2} - \frac{1}{a^2}\right) \\ &\text{and from the second} \\ &\{M + N(B' - A')\}\left(\frac{1}{a'^2} + \frac{1}{b'^2}\right) - 4\pi N\left(\frac{1}{b'^2} - \frac{1}{a'^2}\right) = \Omega\left(\frac{1}{b'^2} - \frac{1}{a'^2}\right) \end{aligned}\right\} \text{(19),}$$

which determine the constants M and N.

152. We shall next investigate the motion of a liquid about an indefinitely thin spherical bowl[1].

Let a be the radius of the sphere of which the bowl forms a part, O its centre, c the radius of the small circle which forms the rim of the bowl, A the pole of this circle which will be called the vertex of the bowl, Q any point on the bowl; also let V be the potential at P of a distribution of matter of density σ on the bowl. Then



$$V = \iint \frac{\sigma dS}{PQ}.$$

Now $\qquad PQ^2 = r^2 + a^2 - 2ar\cos\eta.$

Therefore

$$\frac{dV}{dr} = -\iint \frac{\sigma(r - a\cos\eta)\, dS}{PQ^3},$$

hence

$$-\frac{1}{a}\frac{d(Vr)}{dr} = \iint \frac{\sigma \cos\epsilon dS}{PQ^2},$$

where $\epsilon = \pi - OQP$. The right-hand side of this equation is the magnetic potential at P of a complex magnetic shell of strength σ.

[1] *Proc. Lond. Math. Soc.* vol. XVI. p. 286.

153. Let us now suppose that the motion of an infinite liquid is caused by any system of sources, sinks, or vortex filaments; let Φ be velocity potential due to this system (which we shall call the external system) when the bowl is absent; and let ϕ be the velocity potential after the bowl has been introduced. Then we may put

$$\phi = \Omega + \Phi,$$

where Ω is to be determined.

If the bowl is fixed, which for the present we shall suppose to be the case, the surface condition is

$$-\frac{d\Omega}{dr} = \frac{d\Phi}{dr},$$

when $r = a$. This condition is to be satisfied on both sides of the bowl.

Now, if we remove the bowl, and substitute over its surface a sheet composed of doublets, whose axes are in the directions of the radii passing through them, and whose strength σ, per unit of area, is such that the normal velocity at every point of the sheet is equal and opposite to the normal velocity due to Φ, all the conditions of the problem will be satisfied. But the velocity potential of such a sheet of doublets is analytically equivalent to the magnetic potential of a complex magnetic shell of the same strength, which occupies the position of the bowl, and whose positive side coincides with the sink side of the sheet of doublets; hence the problem is reduced to finding the potential and strength of such a magnetic shell when the normal component of the magnetic force at the surface of the shell is given.

Now we have shown that, if V be the potential of a surface distribution of matter upon the bowl of density σ, then

$$\Omega = -\frac{1}{a}\frac{d(Vr)}{dr};$$

also, if Ω_0 and Ω_i be the values of Ω at two contiguous points just outside and just inside the shell respectively, then

$$\Omega_0 - \Omega_i = 4\pi\sigma.$$

The magnetic force at the surface of the bowl is

$$\begin{aligned} -\frac{d\Omega}{dr} &= \frac{1}{a}\frac{d^2(Vr)}{dr^2} \\ &= -\frac{1}{a^2}\left\{\frac{d}{d\mu}(1-\mu^2)\frac{dV}{d\mu} + \frac{1}{1-\mu^2}\frac{d^2V}{d\psi^2}\right\}, \end{aligned}$$

by Laplace's equation.

Now the value of the magnetic force at the surface of the bowl can always be expanded in a series of spherical surface harmonics Y_n; hence, if

$$-\frac{d\Omega}{dr} = Y_n,$$

$$V = \frac{a^2 Y_n}{n(n+1)};$$

and therefore if

$$-\frac{d\Omega}{dr} = \Sigma_1^\infty Y_n \quad\text{.......................(20)}$$

at the surface, the corresponding value of V at the surface is

$$V = a^2 \Sigma_1^\infty \frac{Y_n}{n(n+1)} \quad\text{.....................(21).}$$

The formula (21) fails when $n = 0$; the only case, however, which is necessary for our purpose to consider, is when the magnetic force is symmetrical with respect to the axis of the bowl, and has a constant value F at its surface. In this case,

$$F = -\frac{d\Omega}{dr}$$

$$= -\frac{1}{a^2}\frac{d}{d\mu}(1-\mu^2)\frac{dV}{d\mu},$$

therefore $$V = \tfrac{1}{2}Fa^2 \log(1-\mu^2) + \tfrac{1}{2}A \log\frac{1+\mu}{1-\mu} + B.$$

Now V must not be infinite when $\mu = 1$, therefore

$$A = Fa^2,$$

and the value of V may be written

$$V = Fa^2 \log a(1+\mu).$$

But, if an infinite straight line extending from the centre of the bowl to $-\infty$ be electrified with line density Fa^2, its potential is

$$= -Fa^2 \log r(1+\mu).$$

Hence V is the potential of the induced charge when the bowl is under the action of a positively electrified line extending from the centre to $-\infty$. If, therefore, χ be the potential of the bowl, under the action of a positive charge of unit intensity, situated at a point on the axis distant u from the centre, and on the negative side of it,

$$V = Fa^2 \int_0^\infty \chi\, du.$$

154. The preceding result enables us to find the velocity potential due to a source situated at the centre of the bowl. In this case

$$\Phi = -\frac{m}{r},$$

therefore
$$-\frac{d\Omega}{dr} = \frac{m}{a^2},$$

therefore
$$\phi = -\frac{m}{a}\int_0^\infty \frac{d(r\chi)}{dr}\,du - \frac{m}{r}.$$

155. *To find the velocity potential due to the motion of the bowl in an infinite liquid.*

(i) Consider the case of motion parallel to the axis.

If the liquid were flowing from right to left past the bowl, the velocity at infinity being equal to w, then

$$\Phi = -wz$$

and
$$\phi = \Omega_z - wz,$$

whence
$$\frac{d\Omega_z}{dr} = w\cos\theta$$

at the surface.

Hence, if the bowl is moving parallel to its axis with velocity u,

$$\phi_3 = \Omega_z.$$

Now, by (21),
$$V_z = -\tfrac{1}{2}wa^2\cos\theta$$

at the surface. V_z is therefore the potential of the induced charge, when the bowl is placed in a uniform field of force parallel to its axis whose potential is $\frac{1}{2}waz +$ const., whence

$$\phi_3 = -\frac{1}{a}\frac{d(V_z r)}{dr}.$$

(ii) Let the bowl be moving perpendicular to its axis with velocity v, and let the plane from which the angle ψ is measured contain the direction of motion; then if ϕ' be the velocity potential,

$$\frac{d\phi'}{dr} = v\cos\psi\sin\theta,$$

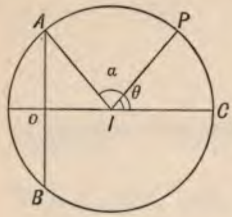
therefore
$$V' = -\tfrac{1}{2}va^2\cos\psi\sin\theta$$

at the surface. V' is therefore the potential of the induced charge, when the bowl is placed in a uniform field of force perpendicular to a plane containing its axis.

(iii) Let the bowl be rotating about an axis.

It is clear that, if the bowl were rotating about an axis through the centre of the sphere of which it forms a part, the bowl would simply cut its way through the liquid without producing any motion. Now, a rotation about any other axis is equivalent to a rotation about a parallel axis through the centre, together with a velocity of translation perpendicular to the plane containing the centre of the bowl, and the original axis of rotation; hence the motion of the liquid due to the rotation of the bowl is equivalent to that due to a properly chosen motion of translation.

156. It thus appears from the preceding articles that the velocity potential due to the motion of the bowl in a liquid, depends upon the electro-static potential of an electrified bowl, which is placed in a field of force whose potential is known. We shall now show how to find this potential, when the field of force is symmetrical with respect to the axis[1].



Let ACB be a section of the bowl through its axis, I the centre of the sphere of which the bowl forms a part, also let $AIC = \alpha$, $PIC = \theta$, $IA = a$, $AB = 2c$.

If in the equation

$$\frac{1}{(1 - 2h\cos\theta + h^2)^{\frac{1}{2}}} = 1 + P_1 h + P_2 h^2 + \ldots\ldots$$

we put $h = \epsilon^{\iota\alpha}$ and equate the real and imaginary parts of the resulting expressions, we obtain

$$\left.\begin{aligned} &\cos\tfrac{1}{2}\alpha + P_1\cos\tfrac{3}{2}\alpha + P_2\cos\tfrac{5}{2}\alpha + \ldots\ldots = \frac{1}{\sqrt{2}\,(\cos\alpha - \cos\theta)^{\frac{1}{2}}} \\ &\sin\tfrac{1}{2}\alpha + P_1\sin\tfrac{3}{2}\alpha + P_2\sin\tfrac{5}{2}\alpha + \ldots\ldots = 0 \end{aligned}\right\} \ldots (22),$$

when $\theta > \alpha$. But if $\theta < \alpha$, the first series is zero, and the second series $= \{2(\cos\theta - \cos\alpha)\}^{-\frac{1}{2}}$.

[1] Ferrers, "On the distribution of electricity on a bowl," *Quart. Journ.* vol. XVIII. p. 97.

Differentiating the second series with respect to α, we obtain

$$\cos\tfrac{1}{2}\alpha + 3P_1\cos\tfrac{3}{2}\alpha + 5\cos\tfrac{5}{2}\alpha + \ldots\ldots\ldots$$
$$= -\frac{\sin\alpha}{\sqrt{2}(\cos\theta - \cos\alpha)^{\frac{3}{2}}} \text{ or } 0 \quad \ldots\ldots(23),$$

according as $\theta <$ or $> \alpha$.

Multiplying (23) by $2\cos\frac{1}{2}(2n+1)\alpha$, we obtain

$$\cos n\alpha + \cos(n+1)\alpha + 3P_1\{\cos(n-1)\alpha + \cos(n+2)\alpha\} + \ldots\ldots$$
$$+ (2n+1)P_n\{1 + \cos(2n+1)\alpha\} + \text{etc.}$$
$$= -\frac{\sqrt{2}\sin\alpha\cos\frac{1}{2}(2n+1)\alpha}{(\cos\theta - \cos\alpha)^{\frac{3}{2}}} \text{ or } 0,$$

according as $\theta <$ or $> \alpha$.

If we suppose $\theta < \alpha$, and integrate both sides with respect to α, between the limits π and α, we shall find that the series

$$\frac{1}{4\pi^2 a}\sum_{s=0}^{s=\infty}(2s+1)\left[\frac{\sin(n-s)\alpha}{n-s} + \frac{\sin(n+s+1)}{n+s+1}\right]P_s$$
$$= \frac{1}{4\pi a}\left[(2n+1)P_n + \frac{\sqrt{2}}{\pi}\int_\alpha^\pi \frac{\sin\alpha\cos\frac{1}{2}(2n+1)\alpha}{(\cos\theta - \cos\alpha)^{\frac{3}{2}}}d\alpha\right] \ldots\ldots(24).$$

But if we suppose $\theta > \alpha$ and integrate with respect to α between the limits α and 0, we shall find that the series in question vanishes. It therefore represents the density of a certain distribution of electricity in the bowl. The potential of this distribution is

$$V = \frac{1}{\pi}\sum_{s=0}^{s=\infty}\left[\frac{\sin(n-s)\alpha}{n-s} + \frac{\sin(n+s+1)\alpha}{n+s+1}\right]\left(\frac{a}{r}\right)^{s+1}P_s \ldots\ldots(25),$$

if $r > a$; but if $r < a$ we must interchange a and r and multiply the result by a/r.

To find the value of V at the surface of the bowl, we must put $r = a$, and differentiate with respect to α; we thus obtain

$$\frac{dV}{d\alpha} = 2\cos\tfrac{1}{2}(2n+1)\alpha\{\cos\tfrac{1}{2}\alpha + P_1\cos\tfrac{1}{2}3\alpha + P_2\cos\tfrac{1}{2}5\alpha + \ldots\ldots\}$$
$$= \frac{\sqrt{2}\cos\frac{1}{2}(2n+1)\alpha}{\pi(\cos\alpha - \cos\theta)^{\frac{1}{2}}} \qquad \theta > \alpha,$$
$$= 0 \qquad \theta < \alpha,$$

by (22).

Hence $$V = \frac{\sqrt{2}}{\pi}\int_0^\alpha \frac{\cos\frac{1}{2}(2n+1)\alpha}{(\cos\alpha - \cos\theta)^{\frac{1}{2}}}\,d\alpha \qquad \theta > \alpha,$$
$$= F(\theta) \qquad \theta < \alpha.$$

To determine $F(\theta)$, let $\alpha = \pi$ in the series (25) for V and we obtain $V = P_n$.

The series on the right-hand side of (25) is the potential of the bowl when placed in a field of force whose potential at the surface of the bowl is equal to $-P_n$, and the density is given by (24); and since the potential of every field of force which is symmetrical with respect to the axis of the bowl can be expanded in a series of zonal harmonics, we can determine the potential and density of the bowl when placed in any such field.

157. In order to obtain the potential when the bowl is placed in a field of force whose potential is $\frac{1}{2}waz$, we must put $n = 1$ in the series (25) and multiply the result by $-\frac{1}{2}wa^2$, hence

$$V_z = -\frac{wa^2}{2\pi}\sum_{s=0}^{s=\infty}\left[\frac{\sin(s-1)\alpha}{s-1} + \frac{\sin(s+2)\alpha}{s+2}\right]\left(\frac{a}{r}\right)^{s+1} P_s \text{......(26).}$$

In order to sum the first series, we have

$$\frac{1}{h^2(1-2h\mu+h^2)^{\frac{1}{2}}} = \frac{1}{h^2} + \frac{P_1}{h} + P_2 + \text{......} P_n h^{n-2} + \text{\&c.},$$

therefore $$\text{const.} - \frac{1}{h} + P_1\log h + P_2 h + \text{......}\frac{P_n h^{n-1}}{n-1} + \text{\&c.}$$
$$= \int\frac{dh}{h^2(1-2h\mu+h^2)^{\frac{1}{2}}}$$
$$= -(1-2\mu h^{-1}+h^{-2})^{\frac{1}{2}} - \mu\sinh^{-1}\frac{h^{-1}-\cos\theta}{\sin\theta}.$$

Putting h successively equal to $a\epsilon^{\iota\alpha}/r$ and $a\epsilon^{-\iota\alpha}/r$, subtracting, and putting S_1 for the first series in (26), we obtain

$$2\iota r^2 a^{-1} S_1 = -(a^2 - 2ar\mu\epsilon^{-\iota\alpha} + r^2\epsilon^{-2\iota\alpha})^{\frac{1}{2}} + (a^2 - ar\mu\epsilon^{\iota\alpha} + r^2c^{2\iota\alpha})^{\frac{1}{2}}$$
$$-\mu a\left[\sinh^{-1}\frac{r\epsilon^{-\iota\alpha} - a\cos\theta}{a\sin\theta} - \sinh^{-1}\frac{r\epsilon^{\iota\alpha} - a\cos\theta}{a\sin\theta}\right]\text{...(27).}$$

Let $$a^2 + r^2\cos 2\alpha - 2ar\cos\alpha\cos\theta = \lambda\cos 2\chi,$$
$$r^2\sin 2\alpha - 2ar\sin\alpha\cos\theta = \lambda\sin 2\chi,$$
$$r^2 + a^2 - 2ar\cos(\alpha-\theta) = p^2,$$
$$r^2 + a^2 - 2ar\cos(\alpha+\theta) = q^2.$$

Then $$\lambda = pq$$
and the first two terms of (27)
$$= 2\iota\sqrt{\lambda}\sin\chi.$$
But
$$4r^2\sin^2\alpha - (p-q)^2 = 2(\lambda - a^2 - r^2\cos 2\alpha + 2ar\cos\alpha\cos\theta)$$
$$= 4\lambda\sin^2\chi.$$
Hence the first two terms
$$= \pm\,\iota\,\{4r^2\sin^2\alpha - (p-q)^2\}^{\frac{1}{2}}.$$
In order to find the value of the last two terms, let us denote the quantity in square brackets by $-2\iota\psi$.

Since
$$\cosh(\sinh^{-1}m - \sinh^{-1}n) = \sqrt{(1+m^2)(1+n^2)} + mn,$$
we easily obtain
$$\begin{aligned} a^2\sin^2\theta\cos 2\psi &= (r^2\epsilon^{2\iota\alpha} - 2ar\epsilon^{\iota\alpha}\cos\theta + a^2)^{\frac{1}{2}} \\ &\qquad\times(r^2\epsilon^{-2\iota\alpha} - 2ar\epsilon^{-\iota\alpha}\cos\theta + a^2)^{\frac{1}{2}} \\ &\quad - (r^2 + a^2\cos^2\theta - 2ar\cos\alpha\cos\theta) \\ &= \lambda - \tfrac{1}{2}(p^2+q^2) + a^2\sin^2\theta, \end{aligned}$$
therefore
$$\psi = \sin^{-1}\frac{q-p}{2a\sin\theta}$$
$$= \sin^{-1}\frac{2r\sin\alpha}{p+q},$$
therefore
$$S_1 = \pm\frac{a}{2r^2}\{4r^2\sin^2\alpha - (p-q)^2\}^{\frac{1}{2}} + \frac{a^2\cos\theta}{r^2}\sin^{-1}\frac{2r\sin\alpha}{p+q}.$$
The second series can be summed in a similar manner, and we shall finally obtain,
$$\begin{aligned} V_z = -\frac{wa}{2\pi}\Big[& r\cos\theta\sin^{-1}\frac{2c}{p+q} \pm \tfrac{1}{2}\{4c^2 - (p-q)^2\}^{\frac{1}{2}} \\ & + \frac{a^3}{r^2}\cos\theta\sin^{-1}\frac{2r\sin\alpha}{p+q} \pm \tfrac{1}{2}\{4r^2\sin^2\alpha - (p-q)^2\}^{\frac{1}{2}}\Big] \quad \text{...(28).} \end{aligned}$$

158. If the positive signs be taken, this is the potential at all points within the space bounded by the plane passing through the rim of the bowl, and that portion of the sphere passing through the centre and rim of the bowl, which lies outside the bowl.

The potential for the space enclosed by the bowl and the plane through its rim is obtained by changing the inverse sine in

the first term to $\pi - \sin^{-1}$, and taking the negative sign before the second term, and the positive sign before the fourth term.

The potential for the remaining portion of space is obtained by changing the inverse sine in the third term to $\pi - \sin^{-1}$, and taking the positive sign before the second term, and the negative sign before the fourth term.

159. We cannot employ an analogous method for determining the potential when the bowl is placed in a field of force perpendicular to a plane containing the axis, since no analytical theorem has been discovered for obtaining the potential of a bowl which is placed in a field of force whose potential is a tesseral harmonic $\sin(m\phi + \epsilon_m) P_n^m(\cos\theta)$.[1]

The solution can however be obtained by the following indirect method. If we put $n = 0$ in (25), and sum the resulting series, we shall obtain the potential of an uninfluenced electrified bowl. Invert the result with respect to a point P in the plane containing the rim of the bowl, whose distance from the centre is equal to f, and multiply the result by $-m$. We shall thus obtain the potential when the bowl is under the influence of a positive charge m at P. Now if we place a negative charge m at a point P' in PO produced such that $OP' = f$, and make the two charges move off to infinity, whilst the product $2m/f^2$ remains constant and equal to $\frac{1}{2}va$, the field of force will ultimately become a uniform field of force perpendicular to a plane containing the axis whose potential is $\frac{1}{2}va \sin\theta \cos\psi$, where ψ is the angle which the plane through the axis and the point (r, θ, ψ) makes with some fixed plane through the axis. The resulting expression for V' will be the potential of the bowl when placed in this field of force.

The result of this process is,

$$V' = -\frac{va}{2\pi}\cos\psi \sin\theta \left[r\sin^{-1}\frac{2c}{p+q} \mp \frac{2cr}{(p+q)^2}\{(p+q)^2 - 4c^2\}^{\frac{1}{2}} \right.$$
$$\left. + \frac{a^3}{r^2}\sin^{-1}\frac{2r\sin\alpha}{p+q} \mp \frac{2a^2c}{r(p+q)^2}\{(p+q)^2 - 4r^2\sin^2\alpha\}^{\frac{1}{2}} \right].$$

[1] If an electrified circular disc is placed in a field of force whose potential is $F(r, \theta)\sin(\phi + \epsilon)$, the potential of the induced charge can be obtained by Bessel's Functions, see *Proc. Camb. Phil. Soc.* vol. v. p. 425; and thence by inversion, we can obtain the potential of an electrified spherical bowl when placed in a field of force of the above form.

The inverse sines and the double signs before the second and fourth terms must be interpreted in the manner explained in the preceding article. (See *Proc. Lond. Math. Soc.* XVI. p. 296.)

The preceding expressions for the velocity potential make the velocity infinite at the edge of the bowl, and therefore the motion represented by the formulae could only be approximately realised in practice.

160. In order to obtain the motion of a liquid in which a solid is moving by means of the velocity potential, it is necessary to find a potential function ϕ which satisfies an equation at the surface of the solid which involves the first derivatives of ϕ, and this circumstance creates a difficulty which has proved insuperable, excepting in the case of an ellipsoid, an anchor ring[1], and a spherical bowl. But if the solid is one of revolution which is moving parallel to its axis, the motion can be determined by means of Stokes' current function, which Rankine[2] has shown has a definite value at the surface of the solid.

Taking the axis of z as the axis of revolution, let w, u be the velocities of the liquid parallel and perpendicular to the axis of z; the surface condition is

$$lw + mu = lV,$$

where V is the velocity of the solid, or

$$\frac{1}{\varpi}\frac{d\psi}{d\varpi}\frac{d\varpi}{ds} + \frac{1}{\varpi}\frac{d\psi}{dz}\frac{dz}{ds} = V\frac{d\varpi}{ds}.$$

Integrating along a meridian curve, we obtain

$$\psi = \tfrac{1}{2}V\varpi^2 \qquad\qquad (29).$$

Now ψ satisfies the equation

$$\frac{d^2\psi}{dz^2} + \frac{d^2\psi}{d\varpi^2} - \frac{1}{\varpi}\frac{d\psi}{d\varpi} = 0.$$

In this put $\psi = \chi\varpi$, and we obtain

$$\frac{d^2\chi}{dz^2} + \frac{d^2\chi}{d\varpi^2} + \frac{1}{\varpi}\frac{d\chi}{d\varpi} - \frac{\chi}{\varpi^2} = 0,$$

which shows that $\chi \sin\phi$ is a solution of Laplace's equation; hence (29) may be written

$$\chi \sin\phi = \tfrac{1}{2}Vy.$$

[1] Hicks, "On Toroidal Functions." *Phil. Trans.* 1881, p. 609.
[2] *Phil. Trans.* 1871.

Hence if U be the electric potential of the induced charge, when the solid is placed in a uniform field of force perpendicular to a plane containing the axis and whose potential is $-\frac{1}{2}Vy$, then $U\varpi \operatorname{cosec} \phi$ will be the current function when the solid is moving with velocity V parallel to its axis.

In the case of a sphere

$$U = \frac{Va^3 y}{2r^3} = \frac{Va^3 \varpi \sin \phi}{2r^3}.$$

Therefore $$\psi = \frac{Va^3 \varpi^2}{2r^3} = \frac{Va^3 \sin^2 \theta}{2r}.$$

EXAMPLES.

1. An ellipsoidal shell is filled with liquid and rotates uniformly about a given diameter; prove that the path of every particle of liquid relatively to the ellipsoid will be an ellipse whose plane is conjugate to the given diameter; and that every particle will sweep out, about the centre of its elliptic path, equal areas in equal times.

2. Liquid flows past the solid ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$, the velocity at infinity being uniform and parallel to x. Prove that the lines of equal pressure on the surface of the ellipsoid are its curves of intersection with the cone $y^2/b^4 + z^2/c^4 = x^2/A^4$, where A is a variable parameter.

3. Liquid is bounded by the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$. If the surface undergo a uniform torsion about a principal axis, prove that the instantaneous velocity potential is proportional to xyz for the liquid in the interior of the ellipsoid, and to

$$\left\{(b^2 - c^2) \frac{d\phi}{da^2} + (c^2 - a^2) \frac{d\phi}{db^2} + (a^2 - b^2) \frac{d\phi}{dc^2}\right\} xyz,$$

for the external space, where

$$\phi = \int_\lambda^\infty \frac{d\lambda}{\sqrt{\{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)\}}}.$$

4. Prove that the velocity potential due to a source of strength m, placed at a point on the axis of a circular disc and distant f from it, at points on the side of the disc on which the source is situated, is

$$\phi = -\int_f^\infty \frac{dP}{dz}\, df - \frac{m}{(r^2+f^2-2fz)^{\frac{1}{2}}},$$

where P is the potential of the induced charge when the disc is under the action of a charge m, situated at a point on the axis on the other side of the disc, and whose distance from it is f.

5. The ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$ is surrounded by an infinite mass of water and rotates about the axis of x. Prove that the component velocities of any particle of water parallel to the axes will be respectively proportional to

$$\frac{dM}{dz} - \frac{dN}{dy},\quad \frac{dN}{dx} - \frac{dL}{dz},\quad \frac{dL}{dy} - \frac{dM}{dx},$$

where $$L = \int_\lambda^\infty \left\{\left(\frac{b^2}{b^2+\psi} - \frac{c^2}{c^2+\psi}\right)\left(1 - \frac{x^2}{a^2+\psi} - \frac{y^2}{b^2+\psi} - \frac{z^2}{c^2+\psi}\right) - 2\left(\frac{by}{b^2+\psi}\right)^2 + 2\left(\frac{cz}{c^2+\psi}\right)^2\right\}\frac{d\psi}{P},$$

$$M = 2b^2xy\int_\lambda^\infty \frac{d\psi}{(a^2+\psi)(b^2+\psi)P},$$

$$N = -2c^2zx\int_\lambda^\infty \frac{d\psi}{(c^2+\psi)(a^2+\psi)P},$$

where $$P = \sqrt{(a^2+\psi)(b^2+\psi)(c^2+\psi)},$$

and λ is the positive root of the equation

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1.$$

Prove also that if the ellipsoid be filled with water, the values of L, M, N with 0 instead of λ for the inferior limit, will similarly determine the velocity of any internal particle of water.

6. A sphere of radius a which is surrounded by an infinite mass of liquid, is strained uniformly so that e, f, g are the principal components of strain after unit time. Prove that the velocity potential of the initially resulting motion is

$$-\tfrac{1}{3}a^5\left(e\frac{d^2}{dx^2} + f\frac{d^2}{dy^2} + g\frac{d^2}{dz^2}\right)\frac{1}{r} - \frac{a^3(e+f+g)}{3r}.$$

7. A sphere of radius a is surrounded by an infinite mass of liquid. If the surface of the sphere be suddenly moved with normal velocity $eyz + fzx + gxy$, prove that the velocity potential of the resulting initial motion is

$$-a^6(eyz + fzx + gxy)/3r^5,$$

where $$r^2 = x^2 + y^2 + z^2.$$

8. Given that

$$x = a(\cosh\alpha + \cos\beta - \cosh\gamma),$$
$$y = 4a\cosh\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta\sinh\tfrac{1}{2}\gamma,$$
$$z = 4a\sinh\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\beta\cosh\tfrac{1}{2}\gamma,$$

transform the equation of continuity into the form

$$(\cos\beta + \cos\gamma)\frac{d^2\phi}{d\alpha^2} + (\cosh\gamma + \cosh\alpha)\frac{d^2\phi}{d\beta^2} + (\cosh\alpha - \cos\beta)\frac{d^2\phi}{d\gamma^2} = 0,$$

and show that the surfaces for which α, β, γ are constant are confocal paraboloids.

Hence show that the velocity potential for infinite liquid streaming past the fixed hyperbolic paraboloid $\beta = \beta_1$, with velocity V parallel to the axis of x at infinity, is given by

$$\phi = V(x - a\beta\sin\beta_1),$$

and write down the corresponding values of ϕ when the fixed surface is the elliptic paraboloid $\alpha = \alpha_1$, or $\gamma = \gamma_1$.

9. The axes of an ellipsoid which is filled with liquid vary with the time in such a manner that the volume of the ellipsoid remains constant; prove that the velocity potential of the liquid is

$$\phi = \tfrac{1}{2}(\dot{a}x^2/a + \dot{b}y^2/b + \dot{c}z^2/c).$$

10. The axes of an ellipsoid which is surrounded by an unlimited liquid vary with the time in such a manner that the ellipsoid always remains similar to itself; prove that

$$\phi = -\tfrac{1}{6}abc(\dot{a}/a + \dot{b}/b + \dot{c}/c)\int_\lambda^\infty \frac{d\psi}{\sqrt{(a^2+\psi)(b^2+\psi)(c^2+\psi)}}.$$

11. Determine the initial motion of liquid outside an ellipsoid, when component velocities (i) px, py, pz; (ii) pyz, pzx, pxy are imparted to every point of its surface; where p is the perpendicular from the centre on to the tangent plane at x, y, z.

CHAPTER VIII.

ON THE GENERAL EQUATIONS OF MOTION OF A SYSTEM OF SOLID BODIES MOVING IN A LIQUID.

161. WHEN a number of solid bodies are moving in an infinite liquid, the motion of the solids is most easily determined by regarding the solids and liquid as constituting a single dynamical system, and then employing Lagrange's equations. But as the methods and formulae employed are different according as the motion of the liquid is cyclic or acyclic, it will be convenient to consider these two cases separately.

Acyclic Motion.

162. The following notation will be employed; let

$u_m, v_m, w_m; p_m, q_m, r_m$ be the linear and angular velocities respectively of any solid S_m, along and about axes *fixed* in the solid.

$\phi_m', \phi_m'', \phi_m'''; \chi_m', \chi_m'', \chi_m'''$ the velocity potentials of the liquid, when the solid S_m is moving with unit linear and angular velocities respectively along and about axes fixed in S_m, and all the other solids are at rest.

Φ_m the velocity potential due to the motion of S_m when all the other solids are at rest.

Ψ the velocity potential of the whole motion.

M_m the mass of S_m.

From § 142 (1) it follows that

$$\Phi_m = u_m\phi_m' + v_m\phi_m'' + w_m\phi_m''' + p_m\chi_m' + q_m\chi_m'' + r_m\chi_m''' \quad \ldots(1),$$

for at the surface of S_m, $d\Phi_m/dn$ is equal to the normal velocity of S_m, and is zero at the surfaces of each of the other solids; whence also,

$$\Psi = \Sigma\Phi_m \quad \ldots\ldots(2).$$

By § 85 (20) if $\mathfrak{T}$ be the kinetic energy of the liquid

$$\mathfrak{T} = -\tfrac{1}{2}\rho\iint\Psi\frac{d\Psi}{dn}\,dS,$$

where the integration extends over all the solids; whence

$$\mathfrak{T} = -\tfrac{1}{2}\rho\iint\Psi\left(\frac{d\Phi_1}{dn}\,dS_1 + \frac{d\Phi_2}{dn}\,dS_2 + \ldots\right).$$

Substituting the values of Ψ, Φ_1, $\Phi_2 \ldots$ in this equation, it appears that $\mathfrak{T}$ is a homogeneous quadratic function of the velocities. If $\frac{1}{2}(u_m u_m)$, (u_m, v_m) denote the coefficients of u_m^2, $u_m v_m$, &c. we obtain

$$\left.\begin{aligned}
(u_1u_1) &= -\rho\iint\phi_1\frac{d\phi_1}{dn}\,dS_1 \\
2(u_1u_2) &= -\rho\iint\phi_2'\frac{d\phi_1'}{dn}\,dS_1 - \rho\iint\phi_1'\frac{d\phi_2'}{dn}\,dS_2 \\
&= -2\rho\iint\phi_2'\frac{d\phi_1'}{dn}\,dS_1 = -2\rho\iint\phi_1'\frac{d\phi_2'}{dn}\,dS_2 \\
&\&c. \qquad \&c.
\end{aligned}\right\} \ldots(3).$$

These equations at once follow from Green's Theorem, and from the fact that $d\phi_1'/dn$ is zero at the surfaces of all the solids except S_1.

163. If all the solids are free, each solid will possess six degrees of freedom, and its position will therefore be determined by six independent coordinates. The velocities of each solid can be expressed in terms of these generalised coordinates and their time fluxes by means of the ordinary methods of Rigid Dynamics, and the kinetic energy of the liquid will therefore be expressible as a homogeneous quadratic function of the generalised velocities of the solids. The coefficients of the velocities will be functions of the generalised coordinates, and of quantities which determine the form and dimensions of the solids. Their values cannot be found without a knowledge of the velocity potential of the liquid, and they have been determined only in a few cases.

The kinetic energy $\mathfrak{T}'$ of the solids can be found by the usual methods, hence if T be the kinetic energy of the solids and liquid,

$$T = \mathfrak{T} + \mathfrak{T}' \quad\text{..........................(4)},$$

from which it is evident that T is a homogeneous quadratic function of the velocities of the solids.

164. Since the coordinates of individual particles of liquid do not enter into the expression for the kinetic energy, it will be necessary to establish the legitimacy of the employment of Lagrange's equations in the present case. The application of these equations is a particular case of the theory of Ignoration of Coordinates.

Let the position of a dynamical system be determined by means of a system of coordinates $\theta_1, \theta_2 \ldots, \chi_1, \chi_2 \ldots$; and let us suppose that the coordinates χ do not enter into the expression for the potential and kinetic energies. Since

$$\frac{dT}{d\chi} = 0, \qquad \frac{dV}{d\chi} = 0.$$

Lagrange's equation corresponding to χ will be

$$\frac{d}{dt}\frac{dT}{d\dot{\chi}} = 0,$$

whence

$$\frac{dT}{d\dot{\chi}} = \text{const.} = \kappa \quad\text{..............................(5)}.$$

The constant κ is the generalised component of momentum corresponding to $\dot{\chi}$; and there will be as many equations of the type (5) as there are coordinates χ. Now whatever the motion of the system at any particular period may be, it can evidently be produced instantaneously from rest by the application of a system of impulsive forces, which must be equivalent to the momentum of the system at the particular period. If however the motion of the system is such that it could always be produced from rest or destroyed, without the application of the impulse components corresponding to $\dot{\chi}$,—in other words if the velocities $\dot{\chi}$ could be produced or destroyed solely by means of impulsive forces arising from the connections of the system,—all the constants κ will be zero, and (5) becomes

$$\frac{dT}{d\dot{\chi}} = 0 \quad\text{..............................(6)}.$$

By means of (6) all the velocities $\dot{\chi}$ can be eliminated from T;

if T' denote the result of this elimination, then since $\dot{\theta}$ and θ enter into T' through $\dot{\chi}$, we have

$$\frac{dT'}{d\dot{\theta}} = \frac{dT}{d\dot{\theta}} + \frac{dT}{d\dot{\chi}_1}\frac{d\dot{\chi}}{d\dot{\theta}} + \frac{dT}{d\dot{\chi}_2}\frac{d\dot{\chi}_2}{d\dot{\theta}} + \dots$$

$$= \frac{dT}{d\dot{\theta}},$$

by (6). Similarly

$$\frac{dT'}{d\theta} = \frac{dT}{d\theta}.$$

Therefore $$\frac{d}{dt}\frac{dT'}{d\dot{\theta}} - \frac{dT'}{d\theta} = \frac{d}{dt}\frac{dT}{d\dot{\theta}} - \frac{dT}{d\theta} = -\frac{dV}{d\theta},$$

which shows that when $\kappa = 0$, we may employ the modified function T' from which the $\dot{\chi}$'s have been eliminated in forming Lagrange's equations.

Now if the dynamical system consists of a number of moving solids together with the liquid in which they are immersed, and which either extends to infinity or is bounded by fixed solids; and if the motion of the liquid is solely due to that of the solids moving about in it, we have shown in §§ 85 and 89 that its motion will be acyclic and irrotational, and that it could be instantaneously produced or destroyed by means of a proper system of impulsive forces applied to the solids and boundaries alone: also since neither the kinetic nor potential energy contains the coordinates of individual particles of liquid, the preceding investigation shows that the equations of motion may be obtained by forming Lagrange's equations by means of the expression for T given by (4), which contains the coordinates and velocities of the solids alone.

If the momenta κ are not zero, Lagrange's equations in their ordinary form cannot be employed. The modified function which must be used in this case will be determined in § 173.

165. The system of impulsive forces which must be applied to the solids to produce the actual motion at any period, when compounded into a single force and a couple about the line of action of the force, is called by Sir W. Thomson the "Impulse of the Motion."

If all the solids are free and the liquid extends to infinity and is at rest there, the Impulse of the Motion is equal to the momentum of the system; and if no impressed forces are in action, it must be

constant in magnitude and direction throughout the motion. But if the liquid has fixed boundaries, the impulse of the motion is equal to the difference between the momentum of the system, and the impulsive forces arising from the pressures exerted by the fixed boundaries.

When there is circulation and the liquid extends to infinity and is at rest there, the impulse of the motion is equal to the impulse of the forces which must be applied to the solids, together with the impulses which must be applied to the barriers in order to produce the cyclic motion.

166. Let p be the impulsive pressure of the liquid, l_1, m_1, n_1 the direction cosines of the normal to S_1; ξ_1, η_1, ζ_1; λ_1, μ_1, ν_1 the force and couple constituents of the impulse which must be applied to S_1, in order to produce the actual motion from rest, then,

$$\xi_1 = M_1 u_1 + \iint p l_1 dS_1$$
$$= M_1 u_1 - \rho \iint \Psi \frac{d\phi_1'}{dn} dS_1.$$

But
$$\frac{d\mathfrak{T}}{du_1} = (u_1 u_1) u_1 + (u_1 v_1) v_1 + \dots$$
$$= -\rho \iint \Psi \frac{d\phi_1'}{dn} dS_1.$$

$$\left.\begin{aligned} &\text{Therefore} && \xi_1 = \frac{dT}{du_1}, \text{ \&c.} \\ &\text{Similarly} && \lambda_1 = \frac{dT}{dp_1}, \text{ \&c.} \end{aligned}\right\} \quad \dots\dots\dots\dots(7).$$

Since T is a homogeneous quadratic function of the velocities of the solids,

$$2T = u_1 \frac{dT}{du_1} + v_1 \frac{dT}{dv_1} + \dots$$
$$= u_1 \xi_1 + v_1 \eta_1 + \dots$$

Differentiating with respect to ξ_1 on the hypothesis that ξ_1, η_1...... are the independent variables, we obtain

$$2 \frac{dT}{d\xi_1} = u_1 + \xi_1 \frac{du_1}{d\xi_1} + \eta_1 \frac{dv_1}{d\eta_1} + \dots$$

Writing out (7) in full, we obtain

$$\xi_1 = \{M_1 + (u_1 u_1)\} u_1 + (u_1 v_1) v_1 + \dots\dots$$
$$\eta_1 = (u_1 v_1) u_1 + \{M_1 + (v_1 v_1)\} v_1 + \dots\dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

Differentiating these equations with respect to ξ_1, on the supposition that ξ_1, η_1...... are the independent variables, we obtain

$$1 = \{M_1 + (u_1u_1)\} \frac{du_1}{d\xi_1} + (u_1v_1) \frac{dv_1}{d\xi_1},$$

$$0 = (u_1v_1) \frac{du_1}{d\xi_1} + \{M_1 + (v_1v_1)\} \frac{dv_1}{d\xi_1},$$

&c. &c.

Multiplying these equations by u_1, v_1... respectively and adding, we obtain

$$u = \xi_1 \frac{du_1}{d\xi_1} + \eta_1 \frac{dv_1}{d\eta_1} + \dots.$$

Whence $$\frac{dT}{d\xi_1} = u_1, \quad \frac{dT}{d\lambda_1} = p_1 \text{ \&c. } \dots\dots\dots\dots \quad (8).$$

Equations (7) and (8) are well-known dynamical relations.

Kirchhoff's Equations.

167. When a single solid moves in an infinite liquid, the equations of motion may be obtained, as Kirchhoff has shown[1], by expressing in an analytical form the fact that the rates of change of the component linear and angular momenta of the system along and about three rectangular axes fixed in the solid are respectively equal to the components of the impressed forces and couples along and about these axes.

Since we are dealing with a single solid we may drop the suffixes and put ω_1, ω_2, ω_3 for the angular velocities of the solid.

If ξ, η, ζ be the component linear momenta along, and λ, μ, ν be the component angular momenta about three rectangular axes which are moving with angular velocities θ_1, θ_2, θ_3 about themselves, of any dynamical system whatever; and if X, Y, Z and L, M, N be the components parallel to and about the axes of the forces and couples respectively which act upon the system, it is known[2] that the equations of motion of the system are

[1] *Vorles. über Math. Phys.* p. 60.

[2] Hayward, *Trans. Camb. Phil. Soc.* vol. x.; see also Besant's *Dynamics*, § 232.

$$\left.\begin{aligned}
&\dot{\xi}-\eta\theta_3+\zeta\theta_2=X\\
&\dot{\eta}-\zeta\theta_1+\xi\theta_3=Y\\
&\dot{\zeta}-\xi\theta_2+\eta\theta_1=Z\\
&\dot{\lambda}-w\eta+v\zeta-\mu\theta_3+\nu\theta_2=L\\
&\dot{\mu}-u\zeta+w\xi-\nu\theta_1+\lambda\theta_3=M\\
&\dot{\nu}-v\xi+u\eta-\lambda\theta_2+\mu\theta_1=N
\end{aligned}\right\}\text{...............(9),}$$

where u, v, w are the component velocities parallel to the axes, of the origin of coordinates.

Since these equations are true for any dynamical system whatever, they will hold when the motion of the liquid in which the solid is immersed is cyclic or rotational or both; but the analytical expressions for the momenta ξ, η, &c. will depend upon the particular kind of motion of the liquid.

When the motion of the liquid is irrotational and acyclic, the momenta are determined by (7); also if the motion is referred to the principal axes of the solid $\theta_1=\omega_1$, $\theta_2=\omega_2$, $\theta_3=\omega_3$, and the equations of motion become

$$\left.\begin{aligned}
&\frac{d}{dt}\frac{dT}{du}-\omega_3\frac{dT}{dv}+\omega_2\frac{dT}{dw}=X\\
&\frac{d}{dt}\frac{dT}{dv}-\omega_1\frac{dT}{dw}+\omega_3\frac{dT}{du}=Y\\
&\frac{d}{dt}\frac{dT}{dw}-\omega_2\frac{dT}{du}+\omega_1\frac{dT}{dv}=Z\\
&\frac{d}{dt}\frac{dT}{d\omega_1}-w\frac{dT}{dv}+v\frac{dT}{dw}-\omega_3\frac{dT}{d\omega_2}+\omega_2\frac{dT}{d\omega_3}=L\\
&\frac{d}{dt}\frac{dT}{d\omega_2}-u\frac{dT}{dw}+w\frac{dT}{du}-\omega_1\frac{dT}{d\omega_3}+\omega_3\frac{dT}{d\omega_1}=M\\
&\frac{d}{dt}\frac{dT}{d\omega_3}-v\frac{dT}{du}+u\frac{dT}{dv}-\omega_2\frac{dT}{d\omega_1}+\omega_1\frac{dT}{d\omega_2}=N
\end{aligned}\right\}\text{.....(10).}$$

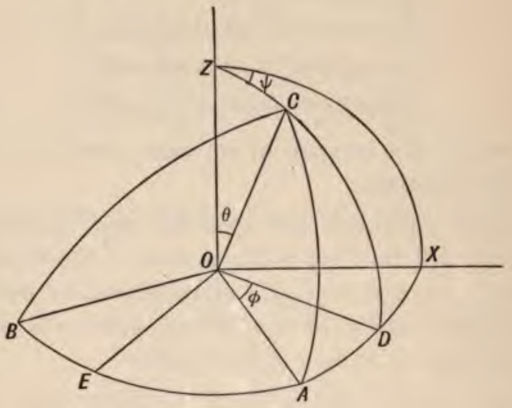
These are Kirchhoff's equations of motion for a single solid moving in an infinite liquid.

Geometrical Equations.

168. We must now express the velocities in terms of the six coordinates, which determine the position of the solid.

Let x, y, z be the coordinates of the centre of inertia O of the solid referred to three *fixed* rectangular axes. Through O

draw OX, OY, OZ parallel to the fixed axes, and let OA, OB, O be the principal axes of the solid at O.



The angular velocities are given by the equations (Routl *Rigid Dynamics*, vol. I. § 256)

$$\left.\begin{aligned}\omega_1 &= \dot\theta \sin\phi - \dot\psi \sin\theta \cos\phi\\ \omega_2 &= \dot\theta\cos\phi + \dot\psi \sin\theta\sin\phi\\ \omega_3 &= \dot\phi + \dot\psi\cos\theta\end{aligned}\right\} \text{..............(11).}$$

Also the component velocity of O in the direction of OD is

$$u\cos\phi - v\sin\phi = (\dot x\cos\psi + \dot y\sin\psi)\cos\theta - \dot z\sin\theta,$$

and in the direction of OE is

$$u\sin\phi + v\cos\phi = -\dot x\sin\psi + \dot y\cos\psi.$$

Solving these equations, and observing that w is the compone velocity of O in the direction of OC, we obtain

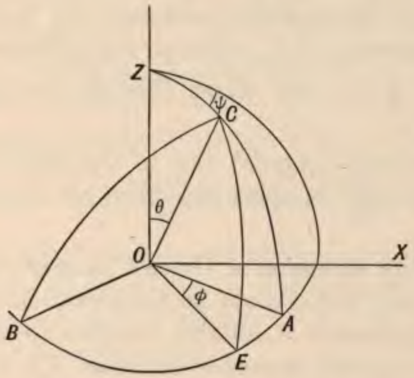
$$\left.\begin{aligned}u = \dot x\,&(\cos\theta\cos\phi\cos\psi - \sin\phi\sin\psi)\\ &+ \dot y\,(\cos\theta\cos\phi\sin\psi + \sin\phi\cos\psi) - \dot z\sin\theta\cos\phi\\ v = -\dot x\,&(\cos\theta\sin\phi\cos\psi + \cos\phi\sin\psi)\\ &- \dot y\,(\cos\theta\sin\phi\sin\psi - \cos\phi\cos\psi) + \dot z\sin\theta\sin\phi\\ w = \dot x\sin\theta&\cos\psi + \dot y\sin\theta\sin\psi + \dot z\cos\theta\end{aligned}\right\} \text{(12).}$$

169. The preceding equations may be considerably simplifi in the case of a solid of revolution.

Let OC be the axis of revolution, OX, OY, OZ three straig lines parallel to axes fixed in space, let w be the velocity of

along OC, u, v the velocities at right angles to OC in and perpendicular to the plane ZOC. Then

$$\left.\begin{aligned} u &= \dot{x}\cos\psi\cos\theta + \dot{y}\sin\psi\cos\theta - \dot{z}\sin\theta \\ v &= -\dot{x}\sin\psi + \dot{y}\cos\psi \\ w &= \dot{x}\cos\psi\sin\theta + \dot{y}\sin\psi\sin\theta + \dot{z}\cos\theta \end{aligned}\right\} \ldots\ldots(13).$$



Also if ω_1, ω_2, ω_3 be the angular velocities about OA, OB, OC

$$\omega_1 = -\dot{\psi}\sin\theta, \quad \omega_2 = \dot{\theta}, \quad \omega_3 = \dot{\phi} + \dot{\psi}\cos\theta \ldots\ldots (14),$$

where the plane COE is fixed in the body.

The velocities of each of the solids can be expressed in a similar manner by means of equations (11) and (12), or (13) and (14); hence if we can obtain the values of the coefficients in terms of the coordinates, the motion can be completely determined.

Cyclic Motion.

170. We must now consider the more general problem of the motion of any number of solids, each one of which has several apertures through which circulation takes place[1].

The following additional notation will be employed. Let

ϕ = velocity potential of the whole motion.

Ψ = do. due to motion of solids alone.

Ω = do. due to cyclic motion.

[1] *Proc. Camb. Phil. Soc.* vol. VI. p. 117.

$\phi_m', \phi_m'', \phi_m'''; \chi_m', \chi_m'', \chi_m'''$, the velocity potentials of the liquid, when the solid S_m is moving with linear and angular velocities respectively along and about axes fixed in S_m, and all the other solids are at rest and there is no circulation.

$\sigma_m, \sigma_m', \sigma_m''\ldots$ the areas of the apertures of S_m.

$\kappa_m, \kappa_m', \kappa_m''\ldots$ the circulations through them.

$\omega_m, \omega_m', \omega_m''\ldots$ the velocity potentials due to unit circulations through the apertures of S_m, when all the solids are at rest.

$\dot{\psi}_m, \dot{\psi}_m', \dot{\psi}_m''\ldots$ the fluxes through the apertures of S_m *relative* to S_m.

Φ_m the velocity potential due to the motion of S_m and the circulations through its apertures, when all the other solids are at rest.

By Thomson's extension of Green's Theorem, it is known that the motion at any period could be instantaneously produced from rest, by the application of suitable impulses to each of the solids, together with uniform impulsive pressures $\kappa_m\rho$, $\kappa'_m\rho\ldots$ applied to every point of the barriers $\sigma_m, \sigma_m'\ldots$ respectively. Let $\bar{X}_m, \bar{Y}_m, \bar{Z}_m$; $\bar{L}_m, \bar{M}_m, \bar{N}_m$ be the force and couple components of the impulse along and about axes fixed in S_m, which must be applied to S_m.

Let $\xi_m, \eta_m, \zeta_m; \lambda_m, \mu_m, \nu_m; \xi_m', \eta_m'\ldots$ be the components of the impulses which must be applied to each of the barriers of S_m; also let $\bar{\xi}_m = \Sigma\xi_m$ &c.; $X_m = \bar{X}_m + \bar{\xi}_m$ &c., and let $\mathfrak{X}_m, \mathfrak{Y}_m, \mathfrak{Z}_m; \mathfrak{L}_m, \mathfrak{M}_m, \mathfrak{N}_m$ be the generalised components corresponding to $u_m, v_m\ldots$ of the momentum of the cyclic motion, when all the solids are at rest.

Let M_m be the mass of S_m, $\mathfrak{T}$ the kinetic energy of the liquid, T that of the whole motion. It will be shown that T is the sum of two homogeneous quadratic functions of the velocities and circulations respectively. Let these be denoted by $\mathfrak{T}$ and $\mathfrak{K}$ respectively, and let $\frac{1}{2}(u_m u_m)$, $(u_m v_m)$ denote the coefficients of $u_m^2, u_m v_m$, &c.

Since the ω's are the velocity potentials due to unit circulations round circuits which cut the apertures to which they correspond once only, when all the solids are at rest, they must satisfy the following conditions.

(i) At all points of the liquid $\nabla^2\omega = 0$, and ω and its first derivatives must be finite and continuous at all points of the liquid, and must vanish at infinity.

(ii) At the surface of each solid $d\omega/dn = 0$.

(iii) ω must be a monocyclic function whose increment is unity for all circuits which cut the barrier to which it corresponds once only, and zero for all circuits which do not cut this barrier.

It therefore follows that

$$\Phi_m = u_m\phi_m' + v_m\phi_m'' + w_m\phi_m''' + p_m\chi_m' + q_m\chi_m'' + r_m\chi_m''' + \kappa_m\omega_m + \kappa_m'\omega_m' + \ldots\ldots\ldots\ldots \quad (15),$$

and that

$$\phi = \Sigma\Phi_m = \Psi + \Omega.$$

171. The kinetic energy of the liquid is

$$\mathfrak{T} = -\tfrac{1}{2}\rho\iint\phi\frac{d\phi}{dn}dS + \tfrac{1}{2}\Sigma\kappa\rho\iint\frac{d\phi}{dn}d\sigma,$$

where the first integral is taken over the surfaces of all the solids, and the second over all the barriers. Since $d\Phi_m/dn$ at the surface of S_m is equal to the normal velocity of S_m, and is zero at the surfaces of each of the other solids,

$$\mathfrak{T} = -\tfrac{1}{2}\rho\iint\Phi\left(\frac{d\Phi_1}{dn}dS_1 + \frac{d\Phi_2}{dn}dS_2 + \ldots\ldots\right) + \tfrac{1}{2}\rho\iint\frac{d\phi}{dn}\left(\kappa_1 d\sigma_1 + \kappa_1' d\sigma_1' + \ldots\ \kappa_2 d\sigma_2 + \ldots\right).$$

We can now show that

$$\left.\begin{aligned}
(u_1u_1) &= -\rho\iint\phi_1'\frac{d\phi_1'}{dn}dS_1, \\
2(u_1u_2) &= -\rho\iint\phi_2'\frac{d\phi_1'}{dn}dS_1 - \rho\iint\phi_1'\frac{d\phi_2'}{dn}dS_2 \\
&= -2\rho\iint\phi_2'\frac{d\phi_1'}{dn}dS_1, \\
2(u_1\kappa_1) &= -\rho\iint\omega_1\frac{d\phi_1'}{dn}dS_1 + \rho\iint\frac{d\phi_1'}{dn}d\sigma_1 = 0, \\
2(u_1\kappa_2) &= -\rho\iint\omega_2\frac{d\phi_1'}{dn}dS_1 + \rho\iint\frac{d\phi_1'}{dn}d\sigma_2 = 0, \\
(\kappa_1\kappa_1) &= \rho\iint\frac{d\omega_1}{dn}d\sigma_1, \\
2(\kappa_1\kappa_1') &= \rho\iint\frac{d\omega_1'}{dn}d\sigma_1 + \rho\iint\frac{d\omega_1}{dn}d\sigma_1' = 2\rho\iint\frac{d\omega_1'}{dn}d\sigma_1,
\end{aligned}\right\} \ldots (16).$$

The above equations can be at once established by Thomson's extension of Green's Theorem. For if in equations (25) and (26)

of § 88, we put $\phi = \omega_1$, $\psi = \phi_1'$, then since ω_1 is a monocyclic function whose increment is unity for all circuits which cut the barrier σ_1 once, and zero for all other circuits, and ϕ_1' is a single valued function, we obtain

$$\iint \omega_1 \frac{d\phi_1'}{dn} dS - \iint \frac{d\phi_1'}{dn} d\sigma_1 = \iint \phi_1' \frac{d\omega_1}{dn} dS.$$

Now $d\phi_1'/dn$ is zero at the surfaces of all the solids except S_1, and $d\omega_1/dn$ is zero at the surfaces of all the solids, whence the third of equations (16) follows at once. The others can be proved in a similar manner; hence the products of velocities and circulations do not enter into the expressions for the kinetic energy of the system, and we may therefore put

$$T = \mathfrak{T} + \mathfrak{K}$$

where $\mathfrak{T}$ is a homogeneous quadratic function of the velocities of the solids alone, and $\mathfrak{K}$ is a similar function of the circulations.

172. If p be the impulsive pressure and l_1, m_1, n_1 the direction cosines of the normal to S_1,

$$\begin{aligned} \bar{X}_1 &= M_1 u_1 + \iint p l_1 dS_1 \\ &= M_1 u_1 - \rho \iint \phi \frac{d\phi_1'}{dn} dS_1. \end{aligned}$$

But
$$\begin{aligned} \frac{d\mathfrak{T}}{du_1} &= (u_1 u_1) u_1 + (u_1 v_1) v_1 + \ldots\ldots \\ &= -\rho \iint (u_1 \phi_1' + v_1 \phi_1'' + \ldots.) \frac{d\phi_1'}{dn} dS_1 \\ &= -\rho \iint \phi \frac{d\phi_1'}{dn} dS_1 + \rho \iint \Omega \frac{d\phi_1'}{dn} dS_1 \\ &= -\rho \iint \phi \frac{d\phi_1'}{dn} dS_1 + \rho \iint \frac{d\phi_1'}{dn} \Sigma (\kappa d\sigma), \end{aligned}$$

where the summation refers to corresponding products, and extends to all the barriers; hence

$$\bar{X}_1 = \frac{dT}{du_1} - \rho \iint \frac{d\phi_1'}{dn} \Sigma (\kappa d\sigma) \ldots\ldots\ldots\ldots\ldots\ldots (17).$$

Also
$$\xi_1 = \kappa_1 \rho \iint l_1 d\sigma, \text{ \&c. \&c.}$$

where l_1, m_1, n_1 are the direction cosines of the normal to the barrier σ_1; whence

$$\bar{\xi}_1 = \Sigma \xi_1 = \rho \iint \Sigma_1 (\kappa l d\sigma),$$

where the summation Σ_1 extends to the barriers of S_1 only; also

$$X_1 = \bar{X}_1 + \bar{\xi}_1 = \frac{dT}{du_1} - \rho \iint \frac{d\phi_1'}{dn} \Sigma (\kappa d\sigma) + \bar{\xi}_1 \text{........(18)}.$$

From (17) we see that the component impulse corresponding to u_1, which must be applied to S_1 in order to keep it at rest, when the cyclic motion is generated by the application of proper impulses to the barriers of all the solids is $-\rho \iint d\phi_1'/dn \,.\, \Sigma (\kappa d\sigma)$; and therefore by (18) the generalised component of momentum $\mathfrak{X}_1$ corresponding to u_1 of the cyclic motion when all the solids are reduced to rest, is

$$\mathfrak{X}_1 = \bar{\xi}_1 - \rho \iint \frac{d\phi_1'}{dn} \Sigma (\kappa d\sigma) = \rho \iint \Sigma_1 (\kappa l d\sigma) - \rho \iint \frac{d\phi_1'}{dn} \Sigma (\kappa d\sigma) \text{..(19)},$$

whence $$X_1 = \frac{dT}{du_1} + \mathfrak{X}_1 \text{..........................(20)}.$$

Similarly it can be shown that

$$L_1 = \frac{dT}{dp_1} + \mathfrak{L}_1 \text{........................(21)},$$

$$\left.\begin{aligned} \text{where} \quad \mathfrak{L}_1 &= \bar{\lambda}_1 - \rho \iint \frac{d\chi_1'}{dn} \Sigma (\kappa d\sigma), \\ \text{and} \quad \bar{\lambda}_1 &= \Sigma \lambda_1 = \rho \iint \Sigma_1 [\kappa (ny - mz) d\sigma] \end{aligned}\right\} \text{..... ......(22)}.$$

173. We must now obtain an expression for the modified Lagrangian function.

Let the coordinates of a dynamical system be divided into two groups θ and χ, the latter of which does not enter into the expression for the energy of the system. Since the kinetic energy is a homogeneous quadratic function of the velocities $\dot{\theta} \ldots \dot{\chi} \ldots$, we may put

$$2T = (\theta\theta)\dot{\theta} + 2(\theta\theta_1)\dot{\theta}\dot{\theta}_1 + \ldots 2(\theta\chi)\dot{\theta}\dot{\chi} + \ldots (\chi\chi)\dot{\chi}^2 + 2(\chi\chi_1)\dot{\chi}\dot{\chi}_1 + \ldots \text{(23)}.$$

In this expression none of the coefficients involve χ, and Lagrange's equation corresponding to χ, gives

$$\frac{dT}{d\dot{\chi}} = \text{const.} = \kappa, \text{ \&c.}$$

where κ is the generalised component of momentum corresponding to $\dot{\chi}$; writing these equations out in full, we obtain

$$\left.\begin{aligned} \kappa &= (\theta\chi)\, \dot{\theta} + (\theta_1\chi)\, \dot{\theta}_1 + \ldots\ldots (\chi\chi)\, \dot{\chi} + (\chi\chi_1)\, \dot{\chi}_1 + \ldots\ldots \\ \kappa_1 &= (\theta\chi_1)\, \dot{\theta} + (\theta_1\chi_1)\, \dot{\theta}_1 + \ldots\ldots (\chi\chi_1)\, \dot{\chi} + (\chi_1\chi_1)\, \dot{\chi}_1 + \ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \text{......(24)},$$

the number of equations being equal to the number of the coordinates χ.

Let $P, P_1, \ldots\ldots$ be the portions of $\kappa, \kappa_1, \ldots\ldots$ which do not involve the $\dot{\chi}$'s, then $P, P_1, \ldots\ldots$ are linear functions of the $\dot{\theta}$'s alone, and (24) may be written

$$\left.\begin{array}{l}(\chi\chi)\,\dot{\chi} + (\chi\chi_1)\,\dot{\chi}_1 + \ldots\ldots = \kappa - P \\ (\chi\chi_1)\,\dot{\chi} + (\chi_1\chi_1)\,\dot{\chi}_1 + \ldots\ldots = \kappa_1 - P_1 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{array}\right\}\ldots\ldots\ldots\ldots(25).$$

If Δ denote the determinant

$$\Delta = \left|\begin{array}{cccc}(\chi\chi), & (\chi\chi_1), & (\chi\chi_2), & \ldots\ldots \\ (\chi\chi_1), & (\chi_1\chi_1), & (\chi_1\chi_2), & \ldots\ldots \\ (\chi\chi_2), & (\chi_1\chi_2), & (\chi_2\chi_2), & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots\end{array}\right|,$$

the solution of (25) may be written

$$\left.\begin{array}{l}\Delta\dot{\chi} = \dfrac{d\Delta}{d(\chi\chi)}(\kappa - P) + \dfrac{d\Delta}{d(\chi\chi_1)}(\kappa_1 - P_1) + \ldots\ldots \\ \Delta\dot{\chi}_1 = \dfrac{d\Delta}{d(\chi\chi_1)}(\kappa - P) + \dfrac{d\Delta}{d(\chi_1\chi_1)}(\kappa_1 - P_1) + \ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{array}\right\}\ldots\ldots(26).$$

If therefore we put

$$(\kappa\kappa) = \frac{1}{\Delta}\frac{d\Delta}{d(\chi\chi)}, \qquad (\kappa\kappa_1) = \frac{1}{\Delta}\frac{d\Delta}{d(\chi\chi_1)}, \text{ \&c.}$$

$$\left.\begin{array}{l}2\mathfrak{P} = (\kappa\kappa)\,P^2 + 2\,(\kappa\kappa_1)\,PP_1 + \ldots \\ 2\mathfrak{K} = (\kappa\kappa)\,\kappa^2 + 2\,(\kappa\kappa_1)\,\kappa\kappa_1 + \ldots\end{array}\right\}\ldots\ldots\ldots(27),$$

(26) may be written

$$\dot{\chi} = \frac{d\mathfrak{K}}{d\kappa} - \frac{d\mathfrak{P}}{dP}, \; \dot{\chi}_1 = \frac{d\mathfrak{K}}{d\kappa_1} - \frac{d\mathfrak{P}}{dP_1} \text{ \&c.} \ldots\ldots\ldots\ldots(28).$$

Let $\mathfrak{T}$ be the portion of T which is independent of $\dot{\chi}$; then, since T is a homogeneous quadratic function of the velocities,

$$\begin{aligned}2T &= \dot{\theta}\frac{dT}{d\dot{\theta}} + \dot{\theta}_1\frac{dT}{d\dot{\theta}_1} + \ldots\ldots\,\dot{\chi}\frac{dT}{d\dot{\chi}_1} + \dot{\chi}_1\frac{dT}{d\dot{\chi}_1} + \ldots\ldots \\ &= \dot{\theta}\frac{d\mathfrak{T}}{d\dot{\theta}} + \dot{\theta}_1\frac{d\mathfrak{T}}{d\dot{\theta}_1} + \ldots\ldots \\ &\quad + \dot{\theta}\,\{(\theta\chi)\,\dot{\chi} + (\theta\chi_1)\,\dot{\chi}_1 + \ldots\ldots\} \\ &\quad + \dot{\theta}_1\,\{(\theta_1\chi)\,\dot{\chi} + (\theta_1\chi_1)\,\dot{\chi}_1 + \ldots\ldots\} + \ldots\ldots \\ &\quad + \dot{\chi}\kappa + \dot{\chi}_1\kappa_1 + \ldots\ldots \\ &= 2\mathfrak{T} + (\kappa + P)\,\dot{\chi} + (\kappa_1 + P_1)\,\dot{\chi}_1 + \ldots\ldots \\ &= 2\mathfrak{T} + 2\mathfrak{K} - 2\mathfrak{P} + \Sigma\left(P\frac{d\mathfrak{K}}{d\kappa} - \kappa\frac{d\mathfrak{P}}{dP}\right).\end{aligned}$$

Writing out the last term in full, it is easily seen from (27) that it vanishes; and therefore since $\mathfrak{P}$ is a homogeneous quadratic function of the velocities $\dot{\theta}$ alone, it follows that T is equal to the sum of a homogeneous quadratic function of the velocities $\dot{\theta}$, together with a similar function of the momenta κ. We may therefore put

$$T = \mathfrak{T} + \mathfrak{K} \quad\text{.........................(29)},$$

where

$$\mathfrak{T} = \mathbf{\mathfrak{T}} - \mathfrak{P} \quad\text{.........................(30)}.$$

Let Θ be the generalised component of momentum corresponding to $\dot{\theta}$, and let $\overline{\Theta}$ be the value of Θ after the velocities $\dot{\theta}$ have been destroyed by means of proper impulses applied to the system. The momenta κ will evidently be unaffected by these impulses, but the velocities $\dot{\chi}$ will be affected, since the impulse required to destroy $\dot{\theta}$ will produce reactions arising from the connections of the system which will change the values of the $\dot{\chi}$'s. Now

$$\Theta = \frac{dT}{d\dot{\theta}} = \frac{d\mathbf{\mathfrak{T}}}{d\dot{\theta}} + (\theta\chi)\dot{\chi} + (\theta\chi_1)\dot{\chi}_1 + \text{.........}$$

$$= \frac{d\mathbf{\mathfrak{T}}}{d\dot{\theta}} + (\theta\chi)\left(\frac{d\mathfrak{K}}{d\kappa} - \frac{d\mathfrak{P}}{dP}\right) + \text{.........}$$

whence

$$\overline{\Theta} = (\theta\chi)\frac{d\mathfrak{K}}{d\kappa} + (\theta\chi_1)\frac{d\mathfrak{K}}{d\kappa_1} + \text{..................} \quad (31),$$

and therefore

$$\frac{dT}{d\dot{\theta}} = \frac{d\mathbf{\mathfrak{T}}}{d\dot{\theta}} + \overline{\Theta} - \Sigma\,(\theta\chi)\frac{d\mathfrak{P}}{dP}$$

$$= \frac{d\mathbf{\mathfrak{T}}}{d\dot{\theta}} + \overline{\Theta} - \Sigma\,\frac{dP}{d\dot{\theta}}\frac{d\mathfrak{P}}{dP}$$

$$= \frac{d\mathfrak{T}}{d\dot{\theta}} + \overline{\Theta} \quad\text{................................(32)},$$

whence

$$\frac{d}{dt}\frac{dT}{d\dot{\theta}} = \frac{d}{dt}\frac{d\mathfrak{T}}{d\dot{\theta}} + \frac{d\overline{\Theta}}{dt} \quad\text{.................. (33)}.$$

It appears from (31) that the momentum $\overline{\Theta}$ is a function of the momenta κ and the coordinates only.

Again

$$\frac{dT}{d\theta} = \frac{d\mathfrak{T}}{d\theta} + \frac{d\mathfrak{K}}{d\theta}.$$

Now since θ enters into $\mathfrak{K}$ through κ, we have

$$\frac{d\mathfrak{K}}{d\theta} = \frac{d\mathfrak{K}}{d\kappa}\frac{d\kappa}{d\theta} + \frac{d\mathfrak{K}}{d\kappa_1}\frac{d\kappa_1}{d\theta} + \dots \frac{\mathfrak{d}\mathfrak{K}}{d\theta},$$

where the symbol $\mathfrak{d}/d\theta$ operates on the coefficients and not on the momenta κ. Differentiating (26) with respect to θ, we obtain

$$0 = (\kappa\kappa)\left(\frac{d\kappa}{d\theta} - \frac{dP}{d\theta}\right) + (\kappa\kappa_1)\left(\frac{d\kappa_1}{d\theta} + \frac{dP_1}{d\theta}\right) + \dots$$
$$+ (\kappa - P)\frac{d}{d\theta}(\kappa\kappa) + (\kappa_1 - P_1)\frac{d}{d\theta}(\kappa\kappa_1) + \dots\dots (34).$$

Multiplying the system of equations of which (34) is the type by $\kappa, \kappa_1 \dots$ respectively and adding, we obtain

$$\frac{d\mathfrak{K}}{d\kappa}\frac{d\kappa}{d\theta} + \frac{d\mathfrak{K}}{d\kappa_1}\frac{d\kappa_1}{d\theta} + \dots\dots$$
$$- \frac{d\mathfrak{K}}{d\kappa}\frac{dP}{d\theta} - \frac{d\mathfrak{K}}{d\kappa_1}\frac{dP_1}{d\theta} - \dots\dots$$
$$+ 2\frac{\mathfrak{d}\mathfrak{K}}{d\theta} - P\frac{\mathfrak{d}}{d\theta}\frac{d\mathfrak{K}}{d\kappa} - P_1\frac{\mathfrak{d}}{d\theta}\cdot\frac{d\mathfrak{K}}{d\kappa_1} - \dots = 0 \quad \dots\dots (35).$$

Multiplying the equations of which (31) is the type by $\dot\theta, \dot\theta_1 \dots$ respectively and adding, we obtain

$$\Sigma(\bar\Theta\dot\theta) = P\frac{d\mathfrak{K}}{d\kappa} + P_1\frac{d\mathfrak{K}}{d\kappa_1} + \dots$$

whence $$\frac{\mathfrak{d}}{d\theta}\Sigma(\bar\Theta\dot\theta) = \Sigma\left(\frac{dP}{d\theta}\frac{d\mathfrak{K}}{d\kappa}\right) + \Sigma\left(P\frac{\mathfrak{d}}{d\theta}\frac{d\mathfrak{K}}{d\kappa}\right),$$

therefore (35) becomes $$\frac{d\mathfrak{K}}{d\theta} + \frac{\mathfrak{d}\mathfrak{K}}{d\theta} - \frac{\mathfrak{d}}{d\theta}\Sigma(\bar\Theta\dot\theta) = 0,$$

whence $$\frac{dT}{d\theta} = \frac{d\mathfrak{T}}{d\theta} - \frac{\mathfrak{d}\mathfrak{K}}{d\theta} + \frac{\mathfrak{d}}{d\theta}\Sigma\,(\bar\Theta\dot\theta).$$

We may now drop the symbol $\mathfrak{d}/d\theta$ on the understanding that the momenta κ are to be treated as constants, and Lagrange's equations become

$$\frac{d}{dt}\frac{d\mathfrak{T}}{d\dot\theta} + \frac{d\Theta}{dt} - \frac{d\mathfrak{T}}{d\theta} + \frac{d\mathfrak{K}}{d\theta} - \frac{d}{d\theta}\Sigma\,(\bar\Theta\dot\theta) + \frac{dV}{d\theta} = 0.$$

Since $\bar\Theta$ and $\mathfrak{K}$ do not contain $\dot\theta$, the modified function is

$$L = \mathfrak{T} + \Sigma\,(\bar\Theta\dot\theta) - \mathfrak{K} + V \dots\dots\dots\dots\dots\dots (36).$$

If the velocities $\dot\theta \dots$ be expressed in terms of new velocities $u \dots$, and $\mathfrak{X}$ be the new momentum corresponding to u after the u's have been destroyed, it can easily be shown that,

$$\Sigma\,(\bar\Theta\dot\theta) = \Sigma\,(\mathfrak{X}u).$$

For let $\dot\theta = Au + A_1u_1 + A_2u_2 + \ldots\ldots$

then $d\dot\theta/du = A, \quad d\dot\theta/du_1 = A_1$, &c.

also by (32), $\Sigma\left(\bar\Theta\frac{d\dot\theta}{du}\right) = \frac{dT^1}{du} - \frac{d\mathfrak{T}}{du} = \mathfrak{X}$,

therefore

$$\Sigma(\mathfrak{X}u) = \bar\Theta\left(u\frac{d\dot\theta}{du} + u_1\frac{d\dot\theta}{du_1} + \right) + \bar\Theta_1\left(u\frac{d\dot\theta_1}{du} + \ldots\ldots\right) + \&c.$$
$$= \Sigma(\bar\Theta\dot\theta),$$

whence (36) may be written

$$L = \mathfrak{T} + \Sigma(\mathfrak{X}u) - \mathfrak{K} + V \ldots\ldots\ldots\ldots\ldots\ldots(37).$$

174. We have therefore obtained a form of Lagrange's equations, which can be employed when the kinetic energy is expressed in terms of the velocities corresponding to the coordinates by which the position of the system is determined, and the constant momenta corresponding to the time fluxes of the ignored coordinates. Now by § 89, when a liquid of density ρ occupies a multiply-connected region, circulation κ can be generated by means of a uniform impulsive pressure $\kappa\rho$ applied to every point of one of the barriers which must be drawn to make the region simply connected, and the circulation thus generated cannot be destroyed excepting by the same process as that by which it has been produced. It therefore appears that the product of the circulation and the density is a quantity in the nature of a generalised component of momentum.

Hence in order to determine the motion of a number of perforated solids in an infinite liquid, we must first calculate by means of (16) the quantities $\mathfrak{T}$ and $\mathfrak{K}$; the former of which is the kinetic energy due to the motion of the solids alone, and is therefore a homogeneous quadratic function of their velocities, and must be expressed in terms of the generalised coordinates and velocities of each solid; and the latter of which is a similar function of the circulations. The quantity $\mathfrak{X}$ in (37) is evidently the generalised component corresponding to u, of the momentum of the cyclic motion which remains after all the solids have been reduced to rest, and its value is given by (19) or (22), according as it is in the nature of a force or a couple.

[1] In this term T is supposed to be expressed in terms of the velocities $u\ldots$ and $\dot\chi\ldots$.

175. We can now ascertain the physical meaning of the generalised velocity $\dot{\chi}$ which corresponds to the momentum $\kappa\rho$.

Let $\dot{\psi}_1$ be the flux through the aperture σ_1 of S_1 relative to S_1. Then if l_1, m_1, n_1 be the direction cosines of the normal to σ_1

$$\dot{\psi}_1 = \iint \left\{ \frac{d\phi}{dn} - l_1(u_1 + q_1 z - r_1 y) - m_1(v_1 + r_1 x - p_1 z) \right.$$
$$\left. - n_1(w_1 + p_1 y - q_1 x) \right\} d\sigma_1$$
$$= \iint \frac{d\Psi}{dn} d\sigma_1 + \iint \frac{d\Omega}{dn} d\sigma_1 - (u_1\xi_1 + v_1\eta_1 + w_1\zeta_1 + p_1\lambda_1 + q_1\mu_1 + r_1\nu_1)/\kappa_1\rho.$$

But
$$\rho \iint \frac{d\Omega}{dn} d\sigma_1 = (\kappa_1\kappa_1)\,\kappa_1 + (\kappa_1\kappa_1')\,\kappa_1' + \ldots\ldots$$
$$= \frac{d\mathfrak{K}}{d\kappa_1}.$$

If therefore we put

$$\alpha_1 = \xi_1/\kappa_1\rho = \iint l_1 d\sigma_1, \quad \beta_1 = \eta_1/\kappa_1\rho, \quad \gamma_1 = \zeta_1/\kappa_1\rho,$$
$$a_1 = \lambda_1/\kappa_1\rho = \iint (n_1 y - m_1 z)\, d\sigma_1, \quad b_1 = \mu_1/\kappa_1\rho, \quad c_1 = \nu_1/\kappa_1\rho,$$

we obtain

$$\dot{\psi}_1 = \iint \frac{d\Psi}{dn} d\sigma_1 - \alpha_1 u_1 - \beta_1 v_1 - \gamma_1 w_1 - a_1 p_1 - b_1 q_1 - c_1 r_1$$
$$+ \frac{1}{\rho}\frac{d\mathfrak{K}}{d\kappa_1} \quad \ldots\ldots\ldots(38).$$

Now if T be expressed as a quadratic function of all the momenta

$$\frac{1}{\rho}\frac{dT}{d\kappa} = \dot{\chi}.$$

But
$$2T = \Sigma u \frac{d\mathfrak{T}}{du} + 2\mathfrak{K} = \Sigma u (X - \mathfrak{X}) + 2\mathfrak{K} \quad \ldots\ldots\ldots (39),$$

by (20). Hence in order to obtain $\dot{\chi}_1$ we must differentiate (39) with respect to κ_1, on the hypothesis that the momenta X are constant, and that u is a function of κ_1; whence by (19) and (22),

$$2\frac{dT}{d\kappa_1} = \Sigma \frac{d\mathfrak{T}}{du}\frac{du}{d\kappa_1} - (\alpha_1 u_1 + \beta_1 v_1 + \gamma_1 w_1 + a_1 p_1 + b_1 q_1 + c_1 r_1)\rho$$
$$+ \rho \iint \frac{d\Psi}{dn} d\sigma_1 + 2\frac{d\mathfrak{K}}{d\kappa_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (40).$$

From (20) we obtain

$$0 = \Sigma\,(u_1v)\frac{dv}{d\kappa_1} + \alpha_1\rho - \rho\iint\frac{d\phi_1'}{dn}\,d\sigma_1,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$0 = \Sigma\,(u_2v)\frac{dv}{d\kappa_1} - \rho\iint\frac{d\phi_2'}{dn}\,d\sigma_2,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

where the summation extends to all the unsuffixed letters including $v = u_1$. Multiplying these equations by $u_1, v_1 \dots$ respectively and adding we obtain

$$\Sigma\frac{d\mathfrak{T}}{du}\frac{du}{d\kappa_1} + (\alpha_1u_1 + \beta_1v_1 + \gamma_1w_1 + a_1p_1 + b_1q_1 + c_1r_1)\,\rho - \rho\iint\frac{d\Psi}{dn}\,d\sigma_1 = 0,$$

whence by (38) and (40) $\qquad \dfrac{dT}{d\kappa_1} = \rho\dot{\psi}_1,$

whence $\qquad \dot{\chi}_1 = \dot{\psi}_1.$

Hence the flux through the aperture σ_1 relative to the solid S_1 is the generalised velocity corresponding to the momentum $\kappa_1\rho$. This theorem was discovered by Sir W. Thomson[1].

176. We shall now apply the preceding results to determine the motion of a single solid having only one aperture.

If u, v, w; $\omega_1, \omega_2, \omega_3$ be the linear and angular velocities of the solid, along and about axes fixed in the solid, and Ω the velocity potential due to the circulation

$$T = \mathfrak{T} + \tfrac{1}{2}K\kappa^2,$$

where $\qquad K = \rho\iint\dfrac{d\Omega}{dn}\,d\sigma.$

Also by (19) and (22)

$$\mathfrak{X} = \kappa\rho\iint\left(l - \frac{d\phi_1}{dn}\right)d\sigma, \ \&c.$$

$$\mathfrak{L} = \kappa\rho\iint\left(ny - mz - \frac{d\chi_1}{dn}\right)d\sigma,$$

$$X = \frac{dT}{du} + \mathfrak{X} \ \&c. \dots\dots\dots\dots(41),$$

$$L = \mathfrak{T} + \mathfrak{X}u + \mathfrak{Y}v + \mathfrak{Z}w + \mathfrak{L}\omega_1 + \mathfrak{M}\omega_2 + \mathfrak{N}\omega_3 - \mathfrak{K} + V \dots(42).$$

[1] *Proc. Roy. Soc. Edin.*, vol. VII. p. 668.

In this case the quantities $\mathfrak{X} \ldots$ are evidently constants, and we can either obtain the motion by expressing $u, v \ldots$ in terms of $\dot{x} \ldots$ by (11) and (12), or by (13) and (14), and then employing Lagrange's equations; or since $X, Y \ldots$ are the components of the momentum of the system along and about the axes of the solid, we may substitute their values in (9) from (41), and thus determine the motion by Kirchhoff's equations.

Motion of a System of Cylinders.

177. If we endeavour to calculate the right-hand side of (37), in the case of the two-dimensional motion of a number of cylinders in an infinite liquid, when there is circulation round some or all the cylinders, it will be found that some of the terms become infinite. In order to avoid this difficulty, we must describe an imaginary fixed circular cylinder in the liquid, the radius of whose cross section is a very large quantity c, and then calculate the value of L for the space contained between the moving cylinders and the outer one, omitting all the terms which vanish when c becomes infinite. It will then be found on substituting the value of L thus obtained in Lagrange's equations and performing the differentiations, that all the terms which become infinite with c disappear, and we thus obtain the equations of motion of the cylinders[1].

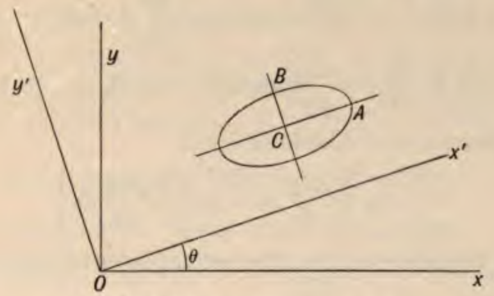
178. The calculation of L can most easily be effected by employing the current function instead of the velocity potential, for the former function is always single valued unless any sources or sinks exist in the liquid.

Let u_1, v_1 be the component velocities of any cylinder S_1 along rectangular axes *fixed in the cylinder*, ω_1 its angular velocity, κ_1 the circulation round any closed circuit which embraces this cylinder once only.

Let the centre O of the cross section of the outer cylinder be the origin, and let x_1, y_1 be the co-ordinates of the centre of inertia of the cross section of S_1 referred to rectangular axes *fixed in space;* x_1', y_1' the co-ordinates of the same point referred to moving axes through O which are parallel to the directions of u_1, v_1. Also let

[1] *Proc. Camb. Phil. Soc.*, vol. VI. p. 135.

χ be the current function, and Ω be the velocity potential of the cyclic motion when all the cylinders are at rest.



In the figure let CA, CB be the axes of any one of the cylinders along which u_1, v_1 are measured, then

$$\mathfrak{X}_1 = \rho \iint \frac{d\chi}{dy'} dx'dy'$$
$$= -\rho \int \chi \frac{dx'}{ds} ds + \rho \left[\int \chi \frac{dx'}{ds} ds \right],$$

where the first integral is to be taken once round the circumference of the cross section of the outer cylinder, and the square brackets denote that the second integral is to be taken once round the circumferences of the cross sections of each of the moving cylinders.

At the surface of each of the moving cylinders χ is constant, hence the second integral vanishes, therefore

$$\mathfrak{X}_1 = -\rho \int \chi \frac{dx'}{ds} ds.$$

Let (r', θ') be polar co-ordinates of a point referred to Ox' as initial line, then at a sufficient distance from O, χ can be expanded in a series of the form

$$\chi = -m \log r' - \frac{1}{r'} (\mathfrak{A}_1 \cos \theta' + \mathfrak{B}_1 \sin \theta') - \dots$$

Therefore

$$\mathfrak{X}_1 = -\rho c \int_0^{2\pi} \left\{ m \log c + \frac{1}{c} (\mathfrak{A}_1 \cos \theta' + \mathfrak{B}_1 \sin \theta') + \dots \right\} \sin \theta' d\theta'$$
$$= -\pi\rho\mathfrak{B}_1 \quad \dots\dots\dots (43).$$

Similarly
$$\mathfrak{Y}_1 = -\rho \iint \frac{d\chi}{dx'} dx'dy' = -\int \chi \frac{dy'}{ds} ds$$
$$= \pi\rho\mathfrak{A}_1 \quad \dots\dots\dots (44).$$

Again, if $\mathfrak{R}_1$ be the angular momentum about C of the cyclic motion,

$$\mathfrak{R}_1 = -\rho\iint\left\{(x'-x_1')\frac{d\chi}{dx'} + (y'-y_1')\frac{d\chi}{dy'}\right\}dx'dy'$$

$$= -\rho\iint\left\{x'\frac{d\chi}{dx'} + y'\frac{d\chi}{dy'}\right\}dx'dy' - \pi\rho\,(\mathfrak{A}_1x_1' + \mathfrak{B}_1y_1').$$

By Stokes' theorem the double integral

$$= -\tfrac{1}{2}\rho\int r^2\frac{d\chi}{dn}\,ds + \tfrac{1}{2}\rho\left[\int r^2\frac{d\chi}{dn}\,ds\right].$$

The first integral $= \pi\rho c^2 m$, the second integral may be written

$$-\tfrac{1}{2}\rho\left[\int r^2 d\Omega/ds\,.\,ds\right],$$

hence

$$\mathfrak{R}_1 = \pi\rho c^2 m - \tfrac{1}{2}\rho\left[\int r^2\frac{d\Omega}{ds}\,ds\right] - \pi\rho\,(\mathfrak{A}_1x_1' + \mathfrak{B}_1y_1') \text{......(45)}.$$

Also

$$2\mathfrak{K} = \rho\int\chi\frac{d\chi}{dn}\,ds - \rho\left[\int\chi\frac{d\chi}{dn}\,ds\right]$$

$$= \rho c\int_0^{2\pi}\chi\frac{d\chi}{dr}\,d\theta + \rho\left[\int\chi\frac{d\Omega}{ds}\,ds\right]$$

$$= \rho c\int_0^{2\pi}\chi\frac{d\chi}{dr}\,d\theta + \rho\Sigma\,(\kappa\chi).$$

The integral

$$= \rho\int_0^{2\pi}\left\{m\log c + \frac{1}{c}(\mathfrak{A}\cos\theta + \mathfrak{B}\sin\theta)\right\}$$

$$\times\left\{m - \frac{1}{c}(\mathfrak{A}\cos\theta + \mathfrak{B}\sin\theta)\right\}d\theta = 2\pi\rho m^2\log c.$$

Whence

$$\mathfrak{K} = \pi\rho m^2\log c + \tfrac{1}{2}\rho\Sigma\,(\kappa\chi) \text{ ....(46)}.$$

Hence we finally obtain

$$L = \mathfrak{T} + \pi\rho\Sigma\,(\mathfrak{A}v - \mathfrak{B}u) + \Sigma\,(\mathfrak{R}\omega)$$

$$- \pi\rho m^2\log c - \tfrac{1}{2}\rho\Sigma\,(\kappa\chi) + V \text{ ......(47)}.$$

If we substitute the preceding expression for L in Lagrange's equations and perform the differentiations, it will be found that the terms $\pi\rho c^2 m$ in $\mathfrak{R}$, and $\pi\rho m^2\log c$ disappear; we may therefore write

$$L = \mathfrak{T} + \pi\rho\Sigma\,(\mathfrak{A}v - \mathfrak{B}u) + \Sigma\,(\mathfrak{R}\omega) - \tfrac{1}{2}\rho\Sigma\,(\kappa\chi) + V \text{ ......(48)}.$$

$$\mathfrak{R} = -\tfrac{1}{2}\rho\left[\int r^2\frac{d\Omega}{ds}\,ds\right] - \pi\rho\,(\mathfrak{A}x' + \mathfrak{B}y') \text{ ......(49)}.$$

179. The quantity $\mathfrak{T}$ which does not depend on the cyclic motion, can be obtained by the ordinary methods. With respect to the other terms we must first obtain the values of χ and Ω; we must then draw from O a series of lines parallel to the directions of u_1, u_2 ..., and take each of these lines successively as the initial line, and expand χ in a series of the form

$$\chi = -m \log r - \frac{1}{r}(\mathfrak{A} \cos\theta + \mathfrak{B} \sin\theta) - \dots$$

which will determine the values of the $\mathfrak{A}$'s and $\mathfrak{B}$'s.

The velocities u, v and the co-ordinates x', y' expressed in terms of x, y, the co-ordinates of C referred to fixed axes, and the angle θ which CA makes with Ox, are given by the equations

$$\left.\begin{aligned} u &= \dot{x}\cos\theta + \dot{y}\sin\theta, & v &= -\dot{x}\sin\theta + \dot{y}\cos\theta \\ x' &= x\cos\theta + y\sin\theta, & y' &= -x\sin\theta + y\cos\theta \end{aligned}\right\} \quad \dots\dots(50).$$

When there are several cylinders, the value of χ at the surfaces of the different cylinders is a function of their forms and positions, and is therefore a function of the co-ordinates; when there is only one cylinder the value of χ at its surface is an absolute constant.

CHAPTER IX.

ON THE MOTION OF A SINGLE SOLID IN AN INFINITE LIQUID.

180. WHEN a single solid is moving in an infinite liquid whose motion is irrotational and acyclic, the kinetic energy of the solid and liquid is a homogeneous quadratic function of the component velocities of the solid alone, and is therefore of the form;

$$\begin{aligned}2T = {} & Pu^2 + Qv^2 + Rw^2 + 2P'vw + 2Q'wu + 2R'uv \\ & + A\omega_1^2 + B\omega_2^2 + C\omega_3^2 + 2A'\omega_2\omega_3 + 2B'\omega_3\omega_1 + 2C'\omega_1\omega_2 \\ & + 2\omega_1(Lu + Mv + Nw) \\ & + 2\omega_2(L'u + M'v + N'w) \\ & + 2\omega_3(L''u + M''v + N''w) \quad \ldots\ldots\ldots\ldots(1),\end{aligned}$$

where u, v, w; $\omega_1, \omega_2, \omega_3$ are the component linear and angular velocities of the solid.

If the motion is referred to the principal axes of the solid, the quantities P, Q, R are called the *effective inertias of the solid parallel to the axes;* and the quantities A, B, C are called the *effective moments of inertia about the axes.* Their values are determined by the equations

$$\left.\begin{aligned} P &= M - \rho\iint\phi_1 l dS \text{ \&c. \&c.} \\ A &= I_1 - \rho\iint\chi_1(ny - mz)\, dS \text{ \&c. \&c.}\end{aligned}\right\}\ldots\ldots\ldots\ldots(2),$$

where M is the mass of the solid, I_1 its moment of inertia about the axis of x, and $\phi_1\ldots$, $\chi_1\ldots$ the constituents of the velocity potential.

The other coefficients depend solely upon the *form* of the solid and the *density* of the liquid; their values are given by § 162. (3).

181. When the form of the solid resembles that of an ellipsoid, which is symmetrical with respect to three perpendicular planes through its centre of inertia, and the motion is referred to the principal axes of the solid at that point, the kinetic energy must remain unchanged when the direction of any one of the component velocities is reversed; hence the kinetic energy cannot contain any of the products of the velocities, and must therefore be of the form;

$$2T = Pu^2 + Qv^2 + Rw^2 + A\omega_1^2 + B\omega_2^2 + C\omega_3^2 \quad \text{......(3)}.$$

If in addition, the solid is one of revolution about the axis of z, the kinetic energy will not be altered if u is changed into v, and ω_1 into ω_2, whence $P = Q$, $A = B$, and

$$2T = P(u^2 + v^2) + Rw^2 + A(\omega_1^2 + \omega_2^2) + C\omega_3^2 \quad \text{......(4)}.$$

Although every solid of revolution must be symmetrical with respect to all planes through its axis, it is not necessarily symmetrical with respect to a plane perpendicular to its axis. The solid formed by the revolution of a cardioid about its axis is an example of such a solid. In this case the kinetic energy will be unaltered when the signs of u, v or ω_3 are changed, and also when u is changed into v and ω_1 into ω_2; hence in this case

$$2T = P(u^2 + v^2) + Rw^3 + A(\omega_1^2 + \omega_2^2) + C\omega_3^2 + 2Nw(\omega_1 + \omega_2) \text{...(5)}.$$

If the solid moves with its axis in one plane, (say zx), v and ω_1 must be zero, and the last term may be got rid of by moving the origin to a point on the axis of z whose distance from the origin is $-N/R$. This point is called the *Centre of Reaction.*

We shall now consider some special cases.

Motion of a Sphere.

182. Let a sphere of radius a, density σ, and mass M be projected with velocity V in an infinite liquid of density ρ; and let the sphere be acted upon by a constant force Z perpendicular to the initial direction of projection.

Let the axis of x be in the direction of projection, and that of z in the direction of the force, then

$$\phi = -\tfrac{1}{2}\frac{a^3}{r^3}(ux + wz)$$

$$2T = P(u^2 + w^2),$$

where
$$P = M - \rho\iint \phi l dS$$
$$= M + \pi\rho a^3 \int_0^\pi \cos^2\theta \sin\theta d\theta$$
$$= M + \tfrac{1}{2}M',$$
where M' is the mass of the liquid displaced. Therefore
$$2T = (M + \tfrac{1}{2}M')(\dot{x}^2 + \dot{z}^2)$$
and Lagrange's equations give
$$\frac{d}{dt}\frac{dT}{d\dot{x}} = 0, \quad \frac{d}{dt}\frac{dT}{d\dot{z}} = Z.$$
Integrating we obtain
$$(M + \tfrac{1}{2}M')\dot{x} = \text{const} = (M + \tfrac{1}{2}M')V$$
whence
$$\dot{x} = V \quad\text{..............................(6)},$$
and
$$(M + \tfrac{1}{2}M')\dot{z} = Zt$$
hence
$$(M + \tfrac{1}{2}M')z = \tfrac{1}{2}Zt^2 \quad\text{... ....(7)}.$$

Since $\dot{x}$ remains constant and equal to its initial value, it follows that if a sphere which is acted upon by no forces, is projected in any direction with given velocity, it will continue to move along that direction with the velocity of projection. The same result can also be shown to be true in the case of any solid which is symmetrical about an axis, and which is projected parallel to that axis. This however is altogether contrary to experience, and the reason of this discrepancy between theory and observation is, that we have assumed the liquid to be frictionless, whereas all liquids with which we are acquainted are more or less viscous. The viscosity gives rise to a retarding force by which the solid and liquid are gradually reduced to rest, and the kinetic energy is converted into heat.

The motion of a sphere in a viscous liquid will be considered in the second volume.

Equation (7) shows that the only effect of the liquid is to produce an apparent increase in the inertia of the sphere, whose amount is equal to half the mass of the liquid displaced.

When the sphere is moving under the action of gravity $Z = (M - M')g$; therefore
$$\dot{z} = \frac{\sigma - \rho}{\sigma + \frac{1}{2}\rho} gt.$$

Hence the sphere will describe a parabola in the liquid with vertical acceleration $g(\sigma - \rho)/(\sigma + \frac{1}{2}\rho)$ [1].

183. In the preceding investigation we have assumed that the liquid always remains in contact with the sphere; but it may happen that the pressure becomes negative at some point of the sphere, in which case a hollow would be formed in the liquid.

If the sphere is moving with constant velocity V in a straight line,

$$\phi = -\frac{Va^3x}{2r^3},$$

also since the origin to which ϕ is referred is in motion with velocity V,

$$\frac{p}{\rho} = \frac{\Pi}{\rho} + V\frac{d\phi}{dx} - \tfrac{1}{2}q^2$$

$$= \frac{\Pi}{\rho} + V^2(\tfrac{9}{8}\cos^2\theta - \tfrac{5}{8})$$

where Π is the pressure at infinity. Hence if

$$\Pi < \tfrac{5}{8}V^2\rho,$$

p will become negative when θ lies between α and $\pi - \alpha$, where $\alpha < \frac{1}{2}\pi$, and a belt of liquid will be thrown off and violently disturbed motion will ensue. For a discussion of the subsequent motion, see a paper by Sir W. Thomson, *Phil. Mag.*, March, 1887.

184. *A sphere of radius* a *and mass* M *is contained within a fixed concentric sphere of radius* c, *and the intervening space is filled with liquid of density* ρ *which is initially at rest. If an impulse* I *be applied to the inner sphere, prove that its initial velocity* w *is equal to*

$$I\left\{M + \frac{2\pi\rho a^3(c^3 + 2a^3)}{3(c^3 - a^3)}\right\}^{-1}.$$

Let $$\phi = \left(\frac{A}{r^2} + Br\right)\cos\theta.$$

Then $$\frac{d\phi}{dr} = w\cos\theta \text{ when } r = a,$$

$$\frac{d\phi}{dr} = 0 \qquad \text{when } r = c.$$

[1] Greenhill, *Mathematical Tripos*, 1877.

Therefore $$A = -\frac{wa^3c^3}{2(c^3-a^3)},\quad B = -\frac{wa^3}{c^3-a^3}$$

and $$\phi = -\frac{wa^3}{c^3-a^3}\left(\frac{c^3}{2r^2}+r\right)\cos\theta.$$

Now if p be the impulsive pressure on inner sphere $p = -\rho\phi$, therefore

$$\begin{aligned} Mw &= I + \rho\iint\phi\cos\theta dS \\ &= I - \frac{\pi\rho wa^3(c^3+2a^3)}{c^3-a^3}\int_0^\pi \cos^2\theta\sin\theta d\theta \\ &= I - \frac{2\pi\rho a^3(c^3+2a^3)w}{3(c^3-a^3)}. \end{aligned}$$

Motion of a Cylinder.

185. When a right circular cylinder is projected in an infinite liquid which is at rest, and no forces are in action, it will move (as will presently be shown) in a straight line with uniform velocity, and the only effect of the liquid will be to produce an apparent increase in the inertia of the cylinder, which is equal to the mass of the liquid displaced. There is however an important difference between the motion of a cylinder and of a sphere, since the space outside a cylinder is a doubly connected space, and hence circulation round the cylinder is possible.

We shall therefore consider the problem in its most general aspect[1].

Let a be the radius of the cylinder, (r, θ) the polar coordinates of any point referred to its centre; (x, y) the coordinates of the same point referred to *fixed* axes, (α, β) the coordinates of the centre of the cylinder, (u, v) its component velocities referred to the *fixed* axes; κ the circulation. Then

$$\begin{aligned} \phi &= -\frac{a^2}{r}(u\cos\theta + v\sin\theta) + \frac{\kappa\theta}{2\pi} \\ &= -a^2\frac{u(x-\alpha)+v(y-\beta)}{(x-\alpha)^2+(y-\beta)^2} + \frac{\kappa}{2\pi}\tan^{-1}\frac{y-\beta}{x-\alpha}, \end{aligned}$$

[1] Lord Rayleigh, "On the irregular flight of a tennis ball," *Mess. Math.*, vol. VII. p. 14; Greenhill, "Note on previous paper," *Mess. Math.*, vol. IX. p. 113.

Now $$\dot{\alpha} = u, \qquad \dot{\beta} = v,$$

whence we easily find

$$\dot{\phi} = -\frac{a^2}{r}(\dot{u}\cos\theta + \dot{v}\sin\theta) + \frac{a^2}{r^2}(u^2 + v^2) - \frac{2a^2}{r^2}(u\cos\theta + v\sin\theta)^2$$
$$+ \frac{\kappa}{2\pi r}(u\sin\theta - v\cos\theta)$$

and therefore when $r = a$

$$\dot{\phi} = -a(\dot{u}\cos\theta + \dot{v}\sin\theta) + u^2 + v^2 - 2(u\cos\theta + v\sin\theta)^2$$
$$+ \frac{\kappa}{2\pi a}(u\sin\theta + v\cos\theta).$$

Also $$q^2 = \left(\frac{d\phi}{dr}\right)^2 + \left(\frac{1}{r}\frac{d\phi}{d\theta}\right)^2.$$

Therefore when $r = a$

$$q^2 = \frac{\kappa^2}{4\pi a^2} + \frac{\kappa}{\pi a}(u\sin\theta - v\cos\theta) + u^2 + v^2.$$

If gravity be the only force in action, and the axis of y be drawn vertically upwards, the pressure is determined by the equation

$$\frac{p}{\rho} - a(\dot{u}\cos\theta + \dot{v}\sin\theta) + \frac{3}{2}(u^2 + v^2) + \frac{\kappa^2}{8\pi a^2} + \frac{\kappa}{\pi a}(u\sin\theta - v\cos\theta),$$
$$- 2(u\cos\theta + v\sin\theta)^2 + g(\beta + a\sin\theta) = \text{const.}$$

Let X, Y be the forces parallel to the axes due to the pressure, then

$$X = -\int_0^{2\pi} ap\cos\theta d\theta, \qquad Y = -\int_0^{2\pi} ap\sin\theta d\theta,$$

whence omitting the terms q/ρ which are independent of θ, and which therefore vanish when integrated round the circle, we obtain

$$X = a\rho\int_0^{2\pi}\left\{\frac{\kappa}{\pi a}(u\sin\theta - v\cos\theta) - a(\dot{u}\cos\theta + \dot{v}\sin\theta)\right.$$
$$\left. - 2(u\cos\theta + v\sin\theta)^2 + ga\sin\theta\right\}\cos\theta d\theta,$$
$$= -\kappa\rho v - \pi a^2\rho\dot{u} \quad \text{......................} (8).$$

Similarly $$Y = \kappa\rho u - \pi a^2\rho\dot{v} + \pi g\rho a^2 \quad \text{..................} (9).$$

Hence if σ be the density of the cylinder, the equations of motion are

$$\pi\sigma a^2\dot{u} = X, \qquad \pi\sigma a^2\dot{v} = Y - \pi\sigma g a^2$$

which by (8) and (9) become

$$\left.\begin{aligned}(\rho+\sigma)\dot{u}+\frac{\kappa\rho}{\pi a^2}v=0\\(\rho+\sigma)\dot{v}-\frac{\kappa\rho}{\pi a^2}u+(\sigma-\rho)g=0\end{aligned}\right\}\text{............(10)}.$$

We draw the following conclusions from (10),

(i) Let $\kappa=0$, $g=0$. In this case the acceleration vanishes and the velocity is constant. Hence if the cylinder is projected with any velocity, it will continue to move along the direction of projection with this velocity, and the only effect of the liquid will be to produce an apparent increase in the inertia of the cylinder which is equal to the mass of the liquid displaced.

(ii) Let $\kappa=0$. In this case the horizontal velocity is constant, and the cylinder will describe a parabola with vertical acceleration

$$g(\sigma-\rho)/(\sigma+\rho).$$

(iii) Let $g=0$: and let the initial velocity be parallel to y and equal to V. Putting $\kappa\rho/\pi a^2(\rho+\sigma)=\lambda$, and integrating (10) we obtain,

$$u=-V\sin\lambda t,\qquad v=V\cos\lambda t,$$
$$\alpha=V\lambda^{-1}\cos\lambda t,\qquad \beta=V\lambda^{-1}\sin\lambda t.$$

If therefore the cylinder is projected with velocity V in any direction, and no external forces are in action, it will describe a circle in the same direction as that of the cyclic motion.

(iv) When neither g nor κ are zero, the integrals of (10) are

$$u=\frac{(\sigma-\rho)g}{(\sigma+\rho)\lambda}-V\sin\lambda t,\qquad v=V\cos\lambda t,$$
$$\alpha=\frac{(\sigma-\rho)gt}{(\sigma+\rho)\lambda}+\frac{V}{\lambda}\cos\lambda t,\qquad \beta=\frac{V}{\lambda}\sin\lambda t,$$

and therefore the cylinder describes a trochoid moving from right to left with mean velocity $(\sigma-\rho)g/(\sigma+\rho)\lambda$.

186. The preceding results may also be obtained by Lagrange's equations; for with the notation of § 178,

$$\mathfrak{T}=\tfrac{1}{2}\pi a^2(\rho+\sigma)(\dot{x}^2+\dot{y}^2),$$

also if (r', θ') be current coordinates

$$\chi=-\frac{\kappa}{2\pi}\log\{(r'\cos\theta'-x)^2+(r'\sin\theta'-y)^2\}^{\frac{1}{2}}$$
$$=-\frac{\kappa}{2\pi}\log r'+\frac{\kappa}{2\pi r'}(x\cos\theta'+y\sin\theta'),$$

whence $\mathfrak{A}=-\kappa x/2\pi,\quad \mathfrak{B}=-\kappa y/2\pi.$

Taking for a moment the origin at the centre of the cylinder, the value of $\mathfrak{R}$ is

$$\mathfrak{R} = -\iint \frac{d\chi}{dr} r^2 dr d\theta$$
$$= \frac{\kappa}{2\pi}\iint r dr d\theta$$
$$= \tfrac{1}{2}\kappa (c^2 - a^2),$$

whence $\mathfrak{R}$ is constant, and the value of L is

$$L = \tfrac{1}{2}\pi a^2 (\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}\kappa\rho (\dot{x}y - \dot{y}x),$$

whence equations (10) at once follow.

187. Let us now suppose that the cross section of the cylinder is any curve, which does not possess cusps projecting into the liquid[1], and that there is no circulation. The kinetic energy will be a homogeneous quadratic function of the velocities u, v, ω, and by changing the directions of the axes we can make the term uv disappear. We shall however for simplicity confine ourselves to the case in which the cross section is a curve (such as an ellipse), which is symmetrical with respect to two perpendicular straight lines through its centre of inertia. In this case all the products will disappear, and

$$2T = Pu^2 + Qv^2 + A\omega^2 \qquad \text{(11)}.$$

Let the liquid be initially at rest, and let the solid be set in motion by means of an impulse F. This impulse is equivalent to a linear impulse F applied at the centre of inertia of the cylinder, together with a couple about its axis. Let Ω be the initial angular velocity due to the couple, β the angle which the direction of the impulse F makes with the initial direction of u.

If θ be the value of this angle at any subsequent time, the Principle of Conservation of Linear Momentum gives,

$$Pu = F\cos\theta, \quad Qv = -F\sin\theta.$$

Substituting in (11) we obtain

$$F^2\left(\frac{\cos^2\theta}{P} + \frac{\sin^2\theta}{Q}\right) + A\dot{\theta}^2 = F^2\left(\frac{\cos^2\beta}{P} + \frac{\sin^2\beta}{Q}\right) + A\Omega^2,$$

or

$$A\dot{\theta}^2 = A\Omega^2 + F^2\left(\frac{1}{P} - \frac{1}{Q}\right)(\sin^2\theta - \sin^2\beta) \qquad \text{(12)}.$$

[1] Greenhill, "On the motion of a cylinder through a frictionless liquid under no forces," *Mess. Math.*, vol. IX. p. 117.

Let $Q > P$; then if

$$\Omega < F \sin\beta \sqrt{\frac{Q-P}{APQ}},$$

$\dot{\theta}$ will vanish, and the cylinder will oscillate; but if

$$\Omega > F \sin\beta \sqrt{\frac{Q-P}{APQ}},$$

$\dot{\theta}$ will never vanish, and the cylinder will make a complete revolution.

Case I. When the cylinder oscillates, (12) may be written

$$\dot{\theta} = I\sqrt{\sin^2\theta - \sin^2\alpha} \qquad (13),$$

where $\quad I = F\sqrt{(Q-P)/APQ},\ I^2 \sin^2\beta - \Omega^2 = I^2 \sin^2\alpha.$

Equation (13) shows that θ can never be $< \alpha$ nor $> \pi - \alpha$ throughout the motion, hence the axis of least effective inertia (i.e. the longest diameter of the cross section) will oscillate between the angles α and $\pi - \alpha$. The cylinder will therefore move so that its flattest side tends to turn itself towards the direction of motion.

Let $$\cos\theta = \cos\alpha \sin\phi,$$

then (13) becomes

$$It = \int_{\frac{1}{2}\pi}^{\phi} \frac{d\phi}{\sqrt{(1 - \cos^2\alpha \sin^2\phi)}},$$

and therefore $$\cos\theta = \cos\alpha \operatorname{sn}(K + It) \qquad (14),$$

and the period of oscillation is $4K/I$.

Let (x, y) be the coordinates of the centre of inertia of the cross section referred to fixed axes, then

$$\dot{x} = u\cos\theta - v\sin\theta,\quad \dot{y} = u\sin\theta + v\cos\theta,$$

whence

$$\left.\begin{aligned} \dot{x} &= \frac{F}{Q} + F\left(\frac{1}{P} - \frac{1}{Q}\right)\cos^2\theta \\ \dot{y} &= F\left(\frac{1}{P} - \frac{1}{Q}\right)\sin\theta\cos\theta \end{aligned}\right\} \qquad (15).$$

These equations show that the centre of inertia of the cross section of the cylinder moves along a straight line parallel to the direction of F with uniform velocity F/Q, superimposed upon which is a variable periodic velocity, and at the same time vibrates

perpendicularly to this line. This kind of motion frequently occurs in hydrodynamics, and a body moving in such a manner is called by Thomson and Tait a *Quadrantal Pendulum*[1].

Substituting the value of θ from (14) in terms of t in (15), and integrating, we shall obtain the values of x and y in terms of t, and the equation of the path will be obtained by eliminating t from the resulting equations.

Case II. When the cylinder makes a complete revolution, let

$$A\Omega^2 + F^2\left(\frac{1}{P} - \frac{1}{Q}\right)\cos^2\beta = \frac{F^2}{k^2}\left(\frac{1}{P} - \frac{1}{Q}\right),$$

then it is easily seen that $k < 1$, and (12) becomes,

$$\dot{\theta} = \frac{I}{k}(1 - k^2\cos^2\theta)^{\frac{1}{2}}$$

whence $$\cos\theta = \mathrm{sn}\,(K - It/k),$$

choosing the constant so that θ vanishes with t. Hence the solution can be continued as before.

Case III. This is the limiting case between I. and II.

Here $$A\Omega^2 = F^2\left(\frac{1}{P} - \frac{1}{Q}\right)\sin^2\beta,$$

and therefore $$\dot{\theta} = I\sin\theta$$

$$It = \log\tan\tfrac{1}{2}\theta.$$

Therefore $$\frac{dy}{d\theta} = \frac{IA}{F}\cos\theta,$$

$$y = \frac{IA}{F}\sin\theta,$$

$$\frac{dx}{d\theta} = \frac{F}{PI}\operatorname{cosec}\theta - \frac{IA}{F}\sin\theta,$$

$$x = \frac{F}{PI}\log\tan\tfrac{1}{2}\theta + \frac{IA}{F}\cos\theta.$$

Putting $IA/F = c$, and eliminating θ we obtain the equation of the path, viz.

$$x = \frac{F}{PI}\log\frac{y}{c + \sqrt{c^2 - y^2}} + \sqrt{c^2 - y^2}.$$

[1] *Natural Philosophy*, vol. I. part I. § 322.

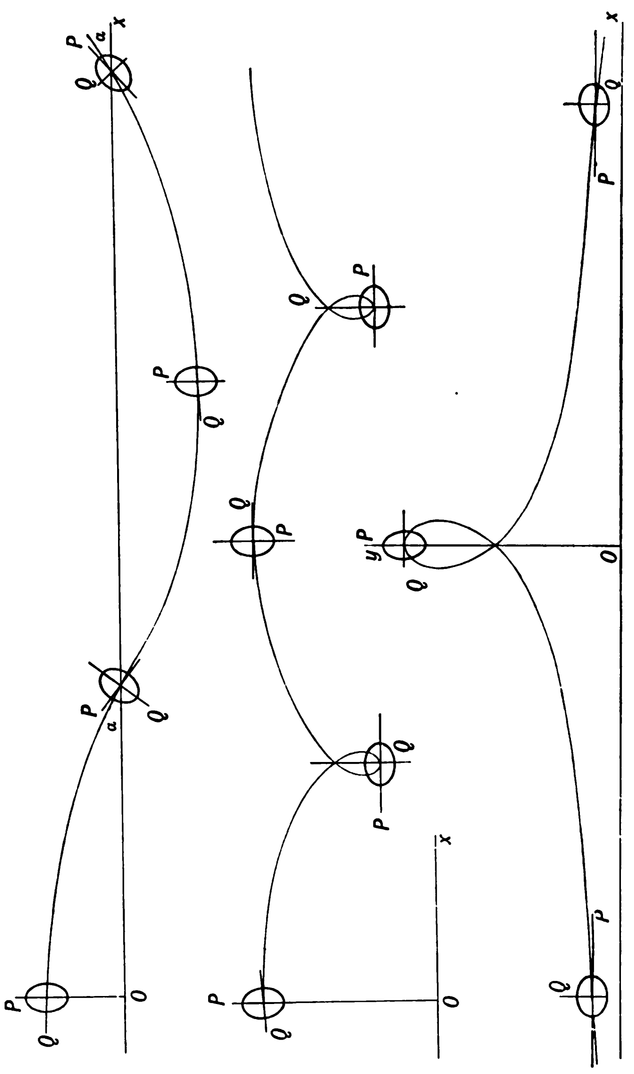


Fig. 1.

Fig. 2.

Fig. 3.

The curves described by the centre of inertia of the cylinder in the three cases, have been traced by Greenhill and are shown in the figures 1, 2, 3 of the accompanying diagram.

If the cylinder is projected parallel to the longest diameter of its cross section and be slightly displaced, it appears from (12) that its motion will be the same as that considered in Case III.

The values of P, Q, A for an elliptic cylinder are,

$$P = M\left(1 + \frac{\rho b}{\sigma a}\right),\ Q = M\left(1 + \frac{\rho a}{\sigma b}\right),$$

$$A = \tfrac{1}{4} M\left\{a^2 + b^2 + \frac{\rho (a^2 - b^2)^2}{2\sigma ab}\right\},$$

whence $Q < P$.

188. If the cross section is a curve such as a cardioid, which is symmetrical with respect to only one straight line through its centre of inertia, which we shall take as the direction of u, the kinetic energy will be

$$2T = Pu^2 + Qv^2 + A\omega^2 + 2L\omega u,$$

and if we transfer the origin to a point on the axis of y whose distance from the origin is $-L/P$, the kinetic energy will be

$$2T = Pu^2 + Qv^2 + \left(A - \frac{L^2}{P}\right)\omega^2,$$

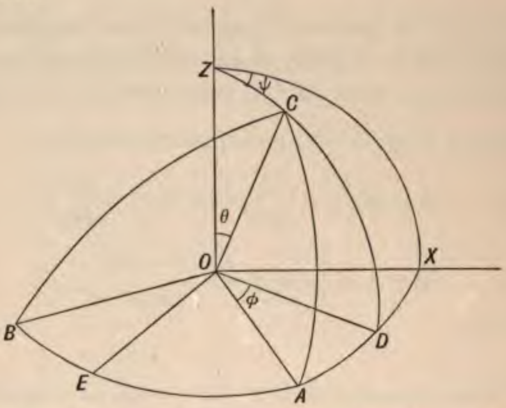
and the previous results apply.

Motion of an Ellipsoid[1].

189. If a solid which is symmetrical with respect to three planes through its centre of inertia, which are mutually at right angles, is set in motion along one of its principal axes, and there are no forces in action, it will continue to move along that direction with uniform velocity. Similarly if it be set in rotation about a principal axis, it will continue to rotate about that axis with uniform angular velocity, provided the solid does not contain any apertures through which circulation takes place.

[1] Greenhill, "Fluid motion between confocal elliptic cylinders and confocal ellipsoids," *Quart. Journ.*, vol. XVI. p. 227.

Let us now suppose that the solid is set in motion by means of an impulse F, whose direction is inclined at an angle α to the axis OC of the solid.



If the axis of z coincides with the direction of the impulse, and no forces are in action, the component momentum parallel to z must be equal to F, and the components parallel to x and y must be zero throughout the motion, whence

$$Pu = -F \sin\theta \cos\phi,$$
$$Qv = F \sin\theta \cos\phi,$$
$$Rw = F \cos\theta.$$

Substituting these values of u, v, w in (3) we obtain,

$$2T = F^2 \left\{\sin^2\theta \left(\frac{\cos^2\phi}{P} + \frac{\sin^2\phi}{Q}\right) + \frac{\cos^2\theta}{R}\right\} + A\omega_1^2 + B\omega_2^2 + C\omega_3^2 \ldots (16).$$

The motion is therefore the same as that of a rigid body rotating about its centre of inertia, under the action of a system of forces whose potential is

$$\tfrac{1}{2} F^2 \left\{\sin^2\theta \left(\frac{\cos^2\phi}{P} + \frac{\sin^2\phi}{Q}\right) + \frac{\cos^2\theta}{R}\right\}.$$

190. Let the solid be moving without rotation along one of its principal axes which coincides with the direction of the axis of x, and be slightly disturbed from its state of steady motion.

Let $u = u_0 + u'$ be the new velocity parallel to x after disturbance. In the beginning of the disturbed motion, u', v, &c. are all small quantities, and Kirchhoff's equations give

$$P\dot{u} = 0, \qquad Q\dot{v} = -Pu_0\omega_3, \qquad R\dot{w} = Pu_0\omega_2,$$
$$A\dot{\omega}_1 = 0, \qquad B\dot{\omega}_2 = (R - P)\, u_0 w, \qquad C\dot{\omega}_3 = (P - Q)\, u_0 v.$$

Hence
$$Q\ddot{v} + \frac{P(P-Q)}{C} u_0^2 v = 0,$$
$$R\ddot{w} + \frac{P(P-R)}{C} u_0^2 w = 0.$$

The motion will therefore be unstable unless P is greater than either Q or R.

191. The only solid for which the quantities P, Q, R, A, B, C have been determined is the ellipsoid.

From § 180 (2),
$$P = M - \rho \iint \phi_1 \, l dS,$$
$$A = I_1 - \rho \iint \chi_1 (ny - mz) \, dS.$$

Hence if we write
$$A' = 2\pi abc \int_0^\infty \frac{d\lambda}{(a^2+\lambda)^{\frac{3}{2}} (b^2+\lambda)^{\frac{1}{2}} (c^2+\lambda)^{\frac{1}{2}}},$$
we obtain from § 147
$$P = M - \rho \iint \frac{A'xl}{A'-4\pi} dS,$$
$$= M - \frac{A'\rho}{A'-4\pi} \iiint dx dy dz,$$
$$= M + \frac{M'A'}{4\pi - A'},$$
by § 7 (9), where M' is the mass of the liquid displaced. Similarly
$$A - I_1 = \tfrac{1}{5} M (b^2 + c^2) - \rho\alpha' \iint yz (ny - mz) \, dS,$$
$$= \tfrac{1}{5} M (b^2 + c^2) - \rho\alpha' \iiint (y^2 - z^2) \, dx dy dz,$$
$$= \tfrac{1}{5} \left\{ M (b^2 + c^2) + \frac{M'(b^2-c^2)(C'-B')}{4\pi(b^2-c^2) + (B'-C')(b^2+c^2)} \right\}.$$

Since $C' > B' > A'$, it follows that $R > Q > P$, whence in the case of the ellipsoid the least axis is the only direction of stable motion.

192. When the motion is such that two of the axes always remain in a plane, the equations of motion can be integrated; for if these axes be the directions of u and v, we have $w = 0$, $\omega_1 = 0$, $\omega_2 = 0$, and
$$2T = Pu^2 + Qv^2 + C\omega_3^2,$$
the kinetic energy is therefore of the same form as in the case of the cylinder considered in § 187.

Under the same circumstances, when the solid is symmetrical with respect to two perpendicular planes through its centre of inertia, the kinetic energy is of the form

$$2T = Pu^2 + Qv^2 + A\omega_3^2 + 2Lu\omega_3,$$

which is reducible to the previous form.

On the Motion of a Solid of Revolution[1].

193. In considering the motion of a solid of revolution, it will be convenient to discuss the case of a ring through whose aperture there is circulation. If in our results we put $\kappa = 0$, we shall obtain the motion of any solid of revolution ring shaped or not when there is no circulation.

Let G be the centre of inertia of a plane curve S, OZ any fixed straight line lying in the plane of S, and let OG be perpendicular to OZ. We shall assume S to be symmetrical with respect to OG, but otherwise it may be of any form, provided there are no singular points capable of giving rise to sharp edges; and the ring will be supposed to be generated by the revolution of S about OZ. Then O will be the centre of inertia of the ring, OZ its axis of unequal moment, which will be called the *axis of the ring;* and the circle described by G will be called the *circular axis of the ring.*

Let the ring be introduced into an infinite liquid which is at rest, and held fixed; let the circular aperture be closed up by means of a plane diaphragm, whose area is σ; and let cyclic irrotational motion be generated by applying to every point of this diaphragm a uniform impulsive pressure $\kappa\rho$, where ρ is the density of the liquid, and then let the diaphragm be removed.

The velocity potential of this cyclic motion will be

$$\phi = \kappa\Omega,$$

where Ω is a monocyclic function whose cyclic constant is unity, and κ is the circulation, round any closed circuit, which embraces the ring once only.

The resultant momentum of the cyclic motion will be parallel

[1] *Proc. Camb. Phil. Soc.*, vol. VI. p. 47.

to the direction of the impulsive pressure on the diaphragm, and equal to $\mathfrak{Z}$; and the energy to $\frac{1}{2}K\kappa^2$, where

$$\mathfrak{Z} = \kappa\rho\sigma - \kappa\rho\iint\Omega n dS,$$
$$K = \rho\iint\frac{d\Omega}{d\nu}\,d\sigma,$$

and n is the z-direction cosine of the normal to the ring drawn outwards, and dS an element of its surface.

If the ring be set in motion, the kinetic energy and momentum of the ring and liquid will be determined by the equations

$$2T = P(u^2+v^2) + Rw^2 + A(\omega_1^2+\omega_2^2) + C\omega_3^2 + K\kappa^2 \text{......(17)},$$

$$\left.\begin{array}{lll} \xi = Pu, & \eta = Bv, & \zeta = Rw + \mathfrak{Z} \\ \lambda = A\omega_1, & \mu = A\omega_2, & \nu = C\omega_3 \end{array}\right\} \text{............(18)}.$$

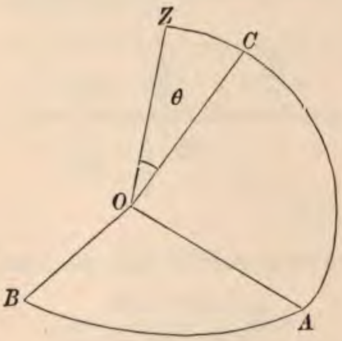
Since the liquid is incapable of producing a couple about the axis of the ring, $\omega_3 = \text{const.} = \omega$ throughout the motion.

Hence, if the ring be let go after the cyclic motion has been generated, it will remain at rest; for the only possible motion will be in the direction of its axis, and consequently

$$2T = Rw^2 + C\omega^2 + K\kappa^2 = \text{its initial value},$$

therefore $$w = 0.$$

194. Let the ring be set in motion by means of an impulsive couple G about any diameter OB of its circular axis.



The axis OC of the ring will evidently move in a fixed plane, which is perpendicular to the axis of the couple. Let θ be the inclination of OC to OZ at time t; u, w the velocities of O along OA and OC.

The principle of Conservation of Linear Momentum gives,

$$-\xi \sin\theta + \zeta \cos\theta = \mathfrak{Z},$$

$$\xi \cos\theta + \zeta \sin\theta = 0,$$

whence

$$\left.\begin{aligned} Pu &= -\mathfrak{Z}\sin\theta \\ Rw &= -\mathfrak{Z}(1-\cos\theta) \end{aligned}\right\} \text{..................} (19).$$

If $\dot{z}$, $\dot{x}$ be the velocities of O along and perpendicular to OZ, then

$$\dot{x} = u\cos\theta + w\sin\theta,$$

$$\dot{z} = w\cos\theta - u\sin\theta.$$

Therefore

$$\left.\begin{aligned} \dot{x} &= \mathfrak{Z}\left(\frac{1}{R}-\frac{1}{P}\right)\sin\theta\cos\theta - \frac{\mathfrak{Z}}{R}\sin\theta \\ \dot{z} &= \frac{\mathfrak{Z}}{P} + \mathfrak{Z}\left(\frac{1}{R}-\frac{1}{P}\right)\cos^2\theta - \frac{\mathfrak{Z}}{R}\cos\theta \end{aligned}\right\} \text{.........}(20).$$

Also $$2T = Pu^2 + Rw^2 + A\dot{\theta}^2 + K\kappa^2 = \text{const.}$$

Substituting the values of u and w from (19) we obtain,

$$A\dot{\theta}^2 = A\omega^2 - \mathfrak{Z}^2\left(\frac{1}{P}+\frac{1}{R}\right) + \frac{2\mathfrak{Z}^2}{R}\cos\theta + \mathfrak{Z}^2\left(\frac{1}{P}-\frac{1}{R}\right)\cos^2\theta \text{...}(21)$$

$$= f(\theta) \text{ say,}$$

where ω is the initial value of $\dot{\theta}$.

The character of the motion depends upon the roots of the equation $f(\theta) = 0$, which we shall now consider.

The roots are

$$\cos\theta = \frac{-\dfrac{\mathfrak{Z}}{R} \pm \sqrt{\left\{\dfrac{\mathfrak{Z}^2}{P^2} - A\omega^2\left(\dfrac{1}{P}-\dfrac{1}{R}\right)\right\}}}{\mathfrak{Z}\left(\dfrac{1}{P}-\dfrac{1}{R}\right)}.$$

Case I. Let $R > P$.

In order that the roots may be real, we must have

$$\omega < \mathfrak{Z}\sqrt{\frac{R}{AP(R-P)}}.$$

If this condition be satisfied, one root will be positive and < 1, and the other will be negative and less than -1. Hence $\dot{\theta}$ will vanish when θ has some value β lying between 0 and $\frac{1}{2}\pi$, and the ring will oscillate between the angles β and $-\beta$.

But if
$$\omega > \mathfrak{Z}\sqrt{\frac{R}{AP(R-P)}}$$
both roots will be imaginary, and $\dot{\theta}$ will never vanish. Hence the ring will make a complete revolution.

Case II. Let $P > R$.

In this case both roots are real, and one of them is positive and < 1 provided ω is sufficiently small; but if ω is sufficiently large both roots will be negative and < -1. In order that one root should not be < -1, it is necessary that
$$\omega < \frac{2\mathfrak{Z}}{\sqrt{AR}}.$$

If this condition be satisfied, the ring will oscillate between the angles β and $-\beta$, where β lies between 0 and π; but if
$$\omega > \frac{2\mathfrak{Z}}{\sqrt{AR}},$$
the ring will make a complete revolution.

195. In order to find the period of oscillation or revolution, as the case may be, we must express θ in terms of t.

Case I. $R > P$.

(i) Let the roots be real and equal to p and $-q$, where
$$q > 1 > p > 0.$$

Equation (21) may be written
$$\dot{\theta}^2 = M^2(\cos\theta - p)(\cos\theta + q),$$
where
$$M^2 = \frac{\mathfrak{Z}^2}{APR}(R-P).$$

Let
$$\cos\theta = \frac{1 - D\cos^2\phi}{1 + D\cos^2\phi},$$
where
$$D = \frac{1-p}{1+p}.$$

Then
$$d\theta = \frac{2\sqrt{D}\sin\phi\, d\phi}{1 + D\cos^2\phi},$$
$$(\cos\theta - p)(\cos\theta + q) = \frac{(1-p)(1+q)}{(1-D\cos^2\phi)^2}(1 - k^2\sin^2\phi),$$
where
$$k^2 = \frac{(q-1)(1-p)}{2(p+q)}.$$

Therefore

$$Mdt = \frac{2}{\sqrt{(1+p)(1+q)}} \frac{d\phi}{\sqrt{(1-k^2\sin^2\phi)}},$$

therefore $$\phi = \operatorname{am} It,$$

where $$I = \tfrac{1}{2}M\sqrt{(1+p)(1+q)}.$$

Therefore

$$\cos\theta = \frac{1+p-(1-p)\operatorname{cn}^2 It}{1+p+(1-p)\operatorname{cn}^2 It},$$

and the period of a complete oscillation is $4K/I$.

(ii) Let the roots be imaginary and equal to $p \pm \iota q$.

Then $$\dot\theta^2 = M^2\{(\cos\theta - p)^2 + q^2\}.$$

Let $$\cos\theta = \frac{1-D+(1+D)\cos\phi}{1+D+(1-D)\cos\phi}.$$

Then $$d\theta = \frac{2\sqrt{D}d\phi}{1+D+(1-D)\cos\phi},$$

and

$$\begin{aligned}\{(\cos\theta-p)^2+q^2\}\{1+D+(1-D)\cos\phi\} &= \{1-D-p(1+D)\}^2+q^2(1+D)^2 \\ &+ 2\cos\phi\,[(1-p)^2+q^2-D^2\{(1+p)^2+q^2\}] + [\{1+D-p(1-D)\}^2 \\ &+ q^2(1-D)^2]\cos^2\phi.\end{aligned}$$

Hence, if $$D^2 = \frac{(1-p)^2+q^2}{(1+p)^2+q^2},$$

the coefficient of $\cos\phi$ will vanish; substituting this value of D, we obtain,

$$\frac{d\theta}{\sqrt{\{(\cos\theta-p)^2+q^2\}}} = \frac{1}{\{(1+p^2+q^2)^2-4p^2\}^{\frac{1}{4}}} \frac{d\phi}{\sqrt{(1-k^2\sin^2\phi}},$$

where $$k^2 = \frac{1}{2}\left[1 + \frac{1-p^2-q^2}{\{(1+p^2+q^2)^2-4p^2\}^{\frac{1}{2}}}\right].$$

Hence $$\phi = \operatorname{am} I't,$$

where $$I' = M\{(1+p^2+q^2)^2-4p^2\}^{\frac{1}{4}};$$

and we finally obtain

$$\tan^2\tfrac{1}{2}\theta = \left\{\frac{(1-p)^2+q^2}{(1+p)^2+q^2}\right\}^{\frac{1}{2}} \cdot \frac{1-\operatorname{cn} I't}{1+\operatorname{cn} I't},$$

and the time of a complete revolution is $4K/I'$.

Case II. $P > R$.

In this case both roots are real, and one root is always negative and numerically greater than unity.

(i) Let the roots be p and $-q$, where $q > 1 > p > 0$. The transformation is the same as in Case I. sub-case (i).

(ii) Let the roots be $-p$ and $-q$, where $q > 1 > p > 0$.

Then $$\dot{\theta}^2 = M^2 (\cos\theta + p)(\cos\theta + q),$$

where $$M^2 = \frac{\mathfrak{Z}^2}{APR}(P - R).$$

In this case we employ the same transformation, but must put

$$D = \frac{1+p}{1-p},$$

$$k^2 = \frac{D(q-1)}{1+q+D(q-1)} = \frac{(q-1)(1-p)}{2(q-p)}.$$

(iii) Let the roots be $-p$ and $-q$, where $q > p > 1$.

We must put $$\cos\theta = \frac{1 - D\sin^2\phi}{1 + D\sin^2\phi},$$

where $$D = -\frac{p-1}{p+1},$$

$$k^2 = \frac{(p-1)(q-1)}{(p+1)(q+1)}.$$

In order to obtain the path described by the centre of inertia O of the ring, we must substitute the value of θ in terms of t in (20), and integrate the result.

We can however ascertain the character of the motion of O without integrating (20). For differentiating (21) we obtain

$$A\ddot{\theta} = -\frac{\mathfrak{Z}^2}{R}\sin\theta - \mathfrak{Z}^2\left(\frac{1}{P} - \frac{1}{R}\right)\sin\theta\cos\theta.$$

Therefore $$\dot{x} = \frac{A\ddot{\theta}}{\mathfrak{Z}},$$

and $$x = \frac{A}{\mathfrak{Z}}(\dot{\theta} - \omega).$$

Also the value of $\dot{z}$ may be written

$$\dot{z} = \frac{\mathfrak{Z}}{P} - \left[\mathfrak{Z}\left(\frac{1}{P} - \frac{1}{R}\right)\cos^2\theta + \frac{\mathfrak{Z}}{R}\cos\theta\right].$$

The term in square brackets has its greatest value when $\theta = 0$, in which case $\dot{z} = 0$; hence $\dot{z}$ can never become negative. The motion of O is such that O moves along the initial direction of the axis of the ring with a uniform velocity, superimposed upon which is a variable periodic velocity; and at the same time vibrates perpendicularly to this line.

196. Since the momentum due to the circulation alone is always perpendicular to the plane of the ring, it follows that if a ring initially at rest be set in motion by means of a couple about a diameter, the direction of this momentum will be changed; and the opposition which the liquid exerts against this action on the part of the ring, will produce a couple tending to oppose the rotation of the ring. Also, since the impressed couple can produce no effect on the linear momentum of the system, it follows that the effect of changing the direction of the momentum due to the circulation, will be to cause the ring to move with a velocity of translation, which gives rise to a linear component of momentum of the whole *system*, such that the resultant of the latter and $\mathfrak{Z}$ (whose direction has been changed) must be a momentum equal to $\mathfrak{Z}$, and whose direction coincides with the original direction of $\mathfrak{Z}$.

197. We shall now investigate the stability of the motion of a ring, which is moving parallel to its axis in the direction of the cyclic motion.

When the motion is steady

$$\zeta = Rw + \mathfrak{Z} = \text{const.} = \gamma,$$
$$\nu = C\omega_3 = \text{const.} = C\Omega,$$
$$\xi = \eta = \lambda = \mu = 0.$$

In order to obtain the disturbed motion, we must have recourse to Kirchhoff's equations of motion; we shall also suppose that the co-ordinate axes are fixed in the ring.

Putting for shortness

$$Z = \gamma + \frac{P(\mathfrak{Z} - \gamma)}{R},$$

the equations of disturbed motion are,

$$P\dot{u} - P\Omega v + \gamma\omega_2 = 0,$$
$$P\dot{v} - \gamma\omega_1 + P\Omega u = 0,$$
$$A\dot{\omega}_1 + Zv + (C - a)\,\Omega\omega_2 = 0,$$
$$A\dot{\omega}_2 - Zu - (C - a)\,\Omega\omega_1 = 0.$$

Putting $u = u'\epsilon^{pt}$, &c. the equation for determining p is,

$$\begin{vmatrix} Pp & -P\Omega & 0 & \gamma \\ P\Omega & Pp & -\gamma & 0 \\ 0 & Z & Ap & (C-A)\Omega \\ -Z & 0 & -(C-A)\Omega & Ap \end{vmatrix} = 0,$$

or

$$A^2P^2p^4 + P[2ZA\gamma + \{(C-A)^2 + A^2\}P\Omega^2]p^2 + \{P(C-A)\Omega^2 + Z\gamma\}^2 = 0.$$

If $Z\gamma$ is positive both values of p^2 are real and negative, and the motion will be stable; but if $Z\gamma$ be negative, the motion will be unstable unless

$$\Omega^2 > -\frac{2AZ\gamma}{P\{(C-A)^2 + A^2\}} \quad \text{..................(22).}$$

If $\Omega = 0$ the roots are

$$p = \pm \iota \sqrt{\frac{Z\gamma}{AP}},$$

and the criterion depends altogether on the sign of $Z\gamma$. Now

$$Z\gamma = \gamma^2 - P\gamma w.$$

(i) Hence if κ and w are both positive, γ will be positive and

$$Z\gamma > 0 \text{ if } R > P,$$

but if $R < P$, $Z\gamma$ will not be positive unless

$$\mathfrak{Z} > (P - R)\,w.$$

(ii) If κ is positive and w negative $= -w'$, $\gamma = \mathfrak{Z} - Rw'$; hence if $\mathfrak{Z} > Rw'$, then $Z\gamma > 0$; but if $\mathfrak{Z} < Rw'$, $Z\gamma$ will not be positive unless

$$(R - P)\,w' > \mathfrak{Z},$$

which requires that $R > P$.

(iii) If $\kappa = 0$ and Ω is not zero,

$$Z\gamma = R\,(R - P)\,w^2.$$

Hence if $R > P$ the motion will be stable; but if $R < P$ the motion will be unstable unless

$$\Omega > w\sqrt{\frac{2AR(P-R)}{P\{(C-A)^2 + A^2\}}}.$$

198. Another kind of steady motion may be obtained by setting the ring in motion by means of a couple G about a diameter of its circular axis, and at the same time applying an impulse $\mathfrak{Z}$ in the opposite direction to that of the cyclic motion.

The effect of the latter impulse is to destroy the linear momentum of the system, hence

$$\xi = 0, \quad \zeta = 0.$$

Therefore
$$u = 0, \quad w = -\frac{\mathfrak{Z}}{R}.$$

Kirchhoff's 5th equation gives

$$\mu = \text{const.} = G = A\dot{\theta}.$$

The motion of the ring is such that its centre of inertia O, describes a circle about a fixed axis parallel to the axis of the couple, through which the plane of the ring always passes. If r be the distance of O from this axis,

$$\frac{\mathfrak{Z}}{R} = -w = r\dot{\theta} = \frac{Gr}{A};$$

therefore
$$r = \frac{A\mathfrak{Z}}{RG}.$$

In order to determine the stability, we must put

$$\xi = Pu, \qquad \eta = Pv, \qquad \zeta = Rw,$$
$$\lambda = A\omega_1, \qquad \mu = G + A\omega_2, \qquad \nu = 0,$$
$$w = -\frac{\mathfrak{Z}}{R} + w, \qquad \omega_2 = -\frac{G}{A} + \omega_2,$$

in Kirchhoff's equations of motion, where the quantities u, v, &c., on the right-hand sides of these equations, are small quantities in the beginning of the disturbed motion. Also, if the axes are fixed in the ring,

$$\theta_1 = \omega_1, \qquad \theta_2 = \frac{G}{A} + \omega_2, \qquad \theta_3 = 0,$$

and the equations of disturbed motion are

$$P\dot{u} + \frac{RG}{A} w = 0,$$
$$P\dot{v} = 0,$$
$$R\dot{w} - \frac{PG}{A} u = 0,$$
$$A\dot{\omega}_1 + \frac{P\mathfrak{Z}}{R} v = 0,$$
$$A\dot{\omega}_2 - \frac{P\mathfrak{Z}}{R} u = 0.$$

From the first and third equations we obtain

$$w = w' \sin\left(\frac{Gt}{A} + \alpha\right),$$

$$u = \frac{Rw'}{P} \cos\left(\frac{Gt}{A} + \alpha\right).$$

The fifth equation gives

$$\omega_2 = \frac{\mathfrak{Z} w'}{G} \sin\left(\frac{Gt}{A} + \alpha\right) + \text{const.}$$

The second and fourth give

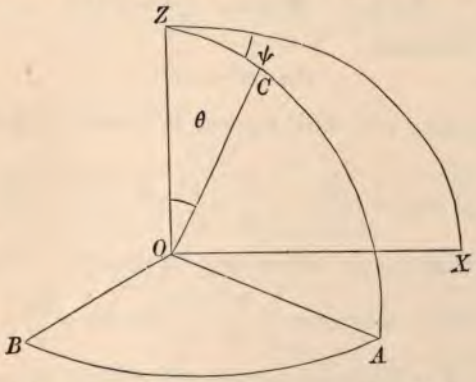
$$v = \text{const.},$$

$$\omega_1 = -\frac{P\mathfrak{Z}}{AR} vt + \text{const.}$$

These equations show that the motion is stable for all displacements which do not tend to remove the centre of inertia from the plane of its motion; but the motion is unstable for all displacements which tend to produce this effect. If the disturbance is such that $v = 0$, the disturbed motion will still be stable, but the axis of rotation will be shifted through a certain angle.

199. A third kind of steady motion, which is helicoidal, is obtained by first communicating to the ring an arbitrary angular velocity Ω about its axis; secondly by applying an impulsive couple G about an axis inclined at an arbitrary angle α to the axis of the ring; and thirdly by applying a determinate impulse in the plane of the axes of the ring and couple.

In order that steady motion may be possible, it is necessary that v and therefore η should be zero throughout the motion. This



condition may be secured by means of an impulsive force whose components in the direction of X and Z are $-\mathfrak{Z} \sin \alpha$, and F.

The equations of momentum are

$$(\xi\cos\theta+\zeta\sin\theta)\cos\psi-\eta\sin\psi=0,$$
$$(\xi\cos\theta+\zeta\sin\theta)\sin\psi+\eta\cos\psi=0,$$
$$-\xi\sin\theta+\zeta\cos\theta=F+\mathfrak{Z}\cos\alpha;$$

whence

$$\left.\begin{aligned}\xi&=-(F+\mathfrak{Z}\cos\alpha)\sin\theta\\ \eta&=0\\ \zeta&=(F+\mathfrak{Z}\cos\alpha)\cos\theta\end{aligned}\right\}\dots\dots\dots\dots\dots\dots(23).$$

Since the components of momentum parallel to the axes of X and Y (which are fixed in direction, but not in position because O is in motion) are zero throughout the motion, the angular momentum about OZ is constant, whence

$$-A\omega_1\sin\theta+C\Omega\cos\theta=G+C\Omega\cos\alpha\dots\dots\dots(24).$$

The equation of energy gives

$$Pu^2+Rw^2+A(\omega_1^{\,2}+\dot\theta^2)=\text{const.},$$

putting $Z=F+\mathfrak{Z}\cos\alpha$, this becomes

$$\frac{Z^2\sin^2\theta}{P}+\frac{\{Z\cos\theta-\mathfrak{Z}\}^2}{R}+\frac{\{G+C\Omega(\cos\alpha-\cos\theta)\}^2}{A\sin^2\theta}$$
$$+A\dot\theta^2=\text{const.}=\text{its initial value}\dots\dots\dots\dots\dots(25).$$

This equation determines the inclination θ of the axis.

200. So far our equations have been perfectly general, we shall now introduce the conditions of steady motion. These are

$$\theta=\alpha,\qquad \dot\psi=\mu,\qquad \ddot\theta=\dot\theta=0\dots\dots\dots\dots\dots(26),$$

whence (24) becomes

$$A\mu\sin^2\alpha=G\dots\dots\dots\dots\dots\dots\dots\dots(27).$$

Differentiating (25) with respect to t, and using (26) and (27), we obtain

$$A\mu^2\cos\alpha-C\Omega\mu+\left(\frac{1}{R}-\frac{1}{P}\right)Z^2\cos\alpha-\frac{Z\mathfrak{Z}}{R}=0\dots(28).$$

In order that steady motion may be possible, we must have

$$C^2\Omega^2>4ZA\cos\alpha\left[\left(\frac{1}{R}-\frac{1}{P}\right)Z\cos\alpha-\frac{\mathfrak{Z}}{R}\right]\dots\dots\dots(29).$$

Hence, if $R>P$ steady motion will always be possible, but if $P>R$, steady motion will be impossible unless the condition (29) is satisfied.

If x, y, z be the co-ordinates of O, we have

$$\dot{x} = (u\cos\theta + w\sin\theta)\cos\psi = \left\{Z\left(\frac{1}{R} - \frac{1}{P}\right)\cos\alpha - \frac{\mathfrak{Z}}{R}\right\}\sin\alpha\cos\mu t,$$

$$\dot{y} = (u\cos\theta + w\sin\theta)\sin\psi = \left\{Z\left(\frac{1}{P} - \frac{1}{R}\right)\cos\alpha - \frac{\mathfrak{Z}}{R}\right\}\sin\alpha\sin\mu t,$$

$$\dot{z} = w\cos\theta - u\sin\theta = Z\left(\frac{\sin^2\alpha}{P} + \frac{\cos^2\alpha}{R}\right) - \frac{\mathfrak{Z}\cos\alpha}{R};$$

whence the centre of inertia describes the helix

$$x = \frac{1}{\mu}\left\{Z\left(\frac{1}{R} - \frac{1}{P}\right)\cos\alpha - \frac{\mathfrak{Z}}{R}\right\}\sin\alpha\sin\mu t,$$

$$y = -\frac{1}{\mu}\left\{Z\left(\frac{1}{R} - \frac{1}{P}\right)\cos\alpha - \frac{\mathfrak{Z}}{R}\right\}\sin\alpha\cos\mu t,$$

$$z = \left\{Z\left(\frac{\sin^2\alpha}{P} + \frac{\cos^2\alpha}{R}\right) - \frac{\mathfrak{Z}\cos\alpha}{R}\right\}t.$$

This last result may be at once obtained from the fact that the impulse of the motion must consist of wrench about a fixed axis[1].

201. To examine the stability differentiate (25) with respect to t, we thus obtain

$$A\ddot{\theta} + f(\theta) = 0.$$

Hence the motion will be stable or unstable according as $f'(\alpha)$ is positive or negative.

Now

$$f(\theta) = \tfrac{1}{2}Z^2\left(\frac{1}{P} - \frac{1}{R}\right)\sin 2\theta + \frac{Z\mathfrak{Z}\sin\theta}{R}$$

$$+ \frac{C\Omega}{A\sin\theta}\{G + C\Omega(\cos\alpha - \cos\theta)\} - \frac{\cos\theta}{A\sin^3\theta}\{G + C\Omega(\cos\alpha - \cos\theta)\}^2;$$

therefore

$$p^2 = f'(\alpha) = A\mu^2(1 + 2\cos^2\alpha)$$

$$- 3C\Omega\mu\cos\alpha + \frac{C^2\Omega^2}{A} - Z^2\left(\frac{1}{R} - \frac{1}{P}\right)\cos 2\alpha + \frac{Z\mathfrak{Z}}{R}\cos\alpha.$$

Eliminating Ω by means of (28) we obtain

$$A^2p^2\mu^2 = A^2\mu^4 + A\mu^2 Z\left\{Z\left(\frac{1}{R} - \frac{1}{P}\right)(1 - 3\cos^2\alpha) + \frac{2\mathfrak{Z}}{R}\cos\alpha\right\}$$

$$+ Z^2\left\{Z\left(\frac{1}{R} - \frac{1}{P}\right)\cos\alpha - \frac{\mathfrak{Z}}{R}\right\}^2.$$

[1] An elementary demonstration of the results of this article when there is no circulation, has been given by Greenhill; *Quart. Jour.*, vol. XVII. p. 86.

The condition that p^2 should be positive is easily found to be that

$$\left\{\frac{2\mathfrak{Z}}{R} + Z\left(\frac{1}{R} - \frac{1}{P}\right)(1 - 3\cos\alpha)\right\}\left\{\frac{2\mathfrak{Z}}{R} - Z\left(\frac{1}{R} - \frac{1}{P}\right)(1 + 3\cos\alpha)\right\}$$

should be positive.

If there is no circulation $\mathfrak{Z} = 0$, $Z = F$, whence the condition becomes

$$F^2\left(\frac{1}{R} - \frac{1}{P}\right)(9\cos^2\alpha - 1) > 0,$$

which requires that α should lie between $\cos^{-1}\frac{1}{3}$ and 0, or between $\pi - \cos^{-1}\frac{1}{3}$ and π.

The azimuthal motion of a solid of revolution when there is no circulation, has been worked out by Prof. Greenhill in the *Quart. Jour.*, vol. XVI. pp. 247—254; and another investigation by him by means of Weierstrass's Functions will be found in the Appendix.

General Motion of a Solid.

202. Having discussed the preceding special cases of motion we shall pass on to discuss certain general theorems relating to the motion of a single solid.

If the form of the solid is similar to that of a two bladed screw propeller of a ship, which is symmetrical when turned through two right angles about the axis of z, the kinetic energy must be unaltered when the signs of u, v, ω_1, ω_2 are all changed, whence

$$2T = Pu^2 + Qv^2 + Rw^2 + 2R'uv + A\omega_1^2 + B\omega_2^2 + C\omega_3^2 + 2C'\omega_1\omega_2$$
$$+ 2\omega_1(Lu + Mv) + 2\omega_2(L'u + M'v) + 2N''\omega_3 w \text{......(30)}.$$

If the solid resembles a four bladed screw propeller which is symmetrical when turned through any multiple of a right angle, the kinetic energy must be unaltered when $-v$, u, $-\omega_2$, ω_1 are written for u, v, ω_1, ω_2 respectively, whence

$$2T = P(u^2 + v^2) + Rw^2 + A(\omega_1^2 + \omega_2^2) + C\omega_3^2$$
$$+ 2L(\omega_1 u + \omega_2 v) + 2M(\omega_1 v - \omega_2 u) + 2N''\omega_3 w \text{......(31)}.$$

In this expression the term $\omega_1 v - \omega_2 u$ can be got rid of by moving the origin along the axis of z.

If the solid is symmetrical with respect to itself when the axes of x and y are turned through any given angle α in either direction, it can be shown that if (1) be transformed by putting

$$u = u' \cos\theta - v' \sin\theta, \qquad \omega_1 = \omega_1' \cos\theta - \omega_2' \sin\theta,$$

$$v = u' \sin\theta + v' \cos\theta, \qquad \omega_2 = \omega_1' \sin\theta + \omega_2' \cos\theta,$$

the condition that the transformed expression for T should be unaltered when θ is put equal to α or $-\alpha$, is that T must be of the form (31).

This kind of symmetry is called helicoidal symmetry.

Let us now suppose that there is another axis situated anywhere, with respect to which the solid possesses helicoidal symmetry. Since the form of (31) is not affected by turning the axes of x and y through any angle, we may suppose them placed so that the other axis of helicoidal symmetry lies in the plane xz. Turning the axes of x and z round that of y through a certain angle ϕ, the new axis of x will be the axis of helicoidal symmetry, and the expression for the energy will be of the form (31) but with the axes of x and z interchanged; whence

$$2T = P(u^2 + v^2 + w^2) + A(\omega_1^2 + \omega_2^2 + \omega_3^2) + 2L(u\omega_1 + v\omega_2 + w\omega_3) \ldots\ldots(32).$$

A solid of this kind is called by Sir W. Thomson an isotropic helicoid[1].

203. When a solid is set in motion along a given direction, it will not in general continue to move along that direction: similarly, if the solid be set in rotation about a given axis, it will not in general continue to rotate about that axis. We shall however show that there are always three directions mutually at right angles, such that if the solid is set in motion along any one of them without rotation and then left to itself, it will continue to move along this direction with uniform velocity.

When there are no impressed forces, Kirchhoff's equations of motion, § 167, are satisfied by putting $\omega_1 = \omega_2 = \omega_3 = 0$, and u, v, w all constant, and

$$\frac{1}{u}\frac{dT}{du} = \frac{1}{v}\frac{dT}{dv} = \frac{1}{w}\frac{dT}{dw},$$

[1] *Proc. Roy. Soc. Edinburgh*, vol. VII. p. 384. See also, Larmor, "On Hydrokinetic Symmetry," *Quart. Journ.* vol. XX. p. 261.

whence $$2T = Pu^2 + Qv^2 + Rw^2 + 2P'vw + 2Q'wu + 2R'uv,$$

and $$\frac{Pu + R'v + Q'w}{u} = \frac{R'u + Qv + P'w}{v} = \frac{Q'u + P'v + Rw}{w}.$$

These equations show that the resultant velocity must be in the direction of one of the principal axes of the ellipsoid

$$Px^2 + Qy^2 + Rw^2 + 2P'yz + 2Q'zx + 2R'xy = \text{const.},$$

which proves the proposition.

204. It is shown in treatises on Statics that every system of forces is reducible to a *wrench;* that is to say a single force, and a couple whose axis coincides with the direction of the force. The ratio of the couple to the force is called the *pitch* of the wrench.

Similarly the motion of every rigid body is reducible to a *twist* about a certain *screw;* that is to say a velocity of translation along a certain line which is called the axis of the screw, together with a rotation about that axis. The ratio of the linear to the angular velocity is called the *pitch* of the screw.

If in § 203 the axes of coordinates coincide with the three directions of permanent translation, the impulse is determined by the equations

$$\xi = \frac{dT}{du} = Pu; \qquad \lambda = \frac{dT}{d\omega_1} = Lu,$$

and therefore consists of a wrench of pitch L/P.

205. The above motion is not the only permanent steady motion of which the solid is capable: for if the velocities and therefore the momenta are constant, Kirchhoff's first three equations of motion give

$$\frac{\xi}{\omega_1} = \frac{\eta}{\omega_2} = \frac{\zeta}{\omega_3} = h \qquad (33),$$

and the last three combined with these give

$$\frac{\lambda - hu}{\omega_1} = \frac{\mu - hv}{\omega_2} = \frac{\nu - hw}{\omega_3} = k \qquad (34).$$

Equation (33) expresses the condition that the axes of the screw and wrench should be parallel, the condition that they should be coincident is

$$\frac{\lambda\omega_1 - \xi u}{\omega_1^2} = \frac{\mu\omega_2 - \eta v}{\omega_2^2} = \frac{\nu\omega_3 - \zeta w}{\omega_3^2},$$

which by (33) is equivalent to (34).

Hence there exists a simply infinite system of possible steady motions, each of which consists of a twist about a certain screw.

The pitches of the screw and the wrench are in general different; if κ be that of the former and κ' that of the latter

$$\kappa' = \frac{\lambda\xi + \mu\eta + \nu\zeta}{\xi^2 + \eta^2 + \zeta^2} = \frac{\omega_1 u + \omega_2 v + \omega_3 w}{\omega_1^2 + \omega_2^2 + \omega_3^2} + \frac{k}{h},$$

whence $$k = h(\kappa' - \kappa).$$

And the expression for the kinetic energy becomes

$$\begin{aligned} 2T &= \xi u + \eta v + \zeta w + \lambda\omega_1 + \mu\omega_2 + \nu\omega_3 \\ &= (\kappa + \kappa')\, h\omega^2, \end{aligned}$$

where ω is the resultant angular velocity.

The values of h and k are not independent, for if the three directions of permanent translation be chosen for the axes of coordinates, and we substitute in (33) and (34) the values of ξ, η, ζ &c. obtained by putting P', Q', R' equal to zero, we shall have the following system of equations

$$\left.\begin{aligned} (A-k)\,\omega_1 + C'\omega_2 + B'\omega_3 + (L-h)\,u + Mv + Nw = 0 \\ \&c. \qquad\qquad \&c. \qquad\qquad \\ (L-h)\,\omega_1 + L'\omega_2 + L''\omega_3 + Pu = 0 \\ \&c. \qquad\qquad \&c. \qquad\qquad \end{aligned}\right\} \text{..(35).}$$

Substituting the values of u, v, w from the last three equations in the first three, it will be found that (35) are of the form

$$\begin{aligned} \alpha\omega_1 + \gamma'\omega_2 + \beta'\omega_3 &= k\omega_1, \\ \gamma'\omega_1 + \beta\omega_2 + \alpha'\omega_3 &= k\omega_2, \\ \beta'\omega_1 + \alpha'\omega_2 + \gamma\omega_3 &= k\omega_3, \end{aligned}$$

whence k is determined by the equation

$$\begin{vmatrix} \alpha - k, & \gamma', & \beta' \\ \gamma', & \beta - k, & \alpha' \\ \beta', & \alpha', & \gamma - k \end{vmatrix} = 0.$$

The roots of this equation are all real; hence to every value of h there are three values of k, which are all real; and the axes of the three screws are mutually at right angles but do not in general intersect.

14—2

206. We shall now show that when the impulse of the motion consists of a couple only, the motion of the solid consists of a motion of translation combined with a motion of rotation, which is the same as that of a certain ellipsoid which rolls upon a certain moveable plane.

Taking the axes of permanent translation as the axes of coordinates, we have $\xi = \eta = \zeta = 0$ throughout the motion; hence

$$Pu + L\omega_1 + L'\omega_2 + L''\omega_3 = 0 \quad \text{\&c.} \quad \text{\&c.}$$

$$A\omega_1 + C'\omega_2 + B'\omega_3 + Lu + Mv + Nw = \lambda \quad \text{\&c.} \quad \text{\&c.}$$

If we eliminate u, v, w from the last three equations by means of the first three, it will be found that

$$\lambda = \frac{d\Theta}{d\omega_1}, \quad \mu = \frac{d\Theta}{d\omega_2}, \quad \nu = \frac{d\Theta}{d\omega_3},$$

where

$$2\Theta = \mathfrak{P}\omega_1^2 + \mathfrak{Q}\omega_2^2 + \mathfrak{R}\omega_3^2 + 2\mathfrak{P}'\omega_2\omega_3 + 2\mathfrak{Q}'\omega_3\omega_1 + 2\mathfrak{R}'\omega_1\omega_2 \ldots (36),$$

$$\mathfrak{P} = A - \frac{L^2}{P} - \frac{M^2}{Q} - \frac{N^2}{R} \text{ \&c., \&c.,}$$

$$\mathfrak{P}' = A' - \frac{L'L''}{P} - \frac{M'M''}{Q} - \frac{N'N''}{R} \text{ \&c., \&c.}$$

The equations of motion are

$$\dot{\lambda} = \omega_3\mu - \omega_2\nu \text{ \&c. \&c.} \ldots\ldots\ldots\ldots\ldots\ldots (37).$$

In equations (37) let us change the directions of the axes which are fixed in the body, so that they coincide with the principal axes of the quadric

$$\mathfrak{P}x^2 + \mathfrak{Q}y^2 + \mathfrak{R}z^2 + 2\mathfrak{P}'yz + 2\mathfrak{Q}'zx + 2\mathfrak{R}'xy = \text{const.}$$

If this be done, and the equation of the quadric referred to these axes is

$$\alpha x^2 + \beta y^2 + \gamma z^2 = 1,$$

we shall have

$$\lambda' = \alpha\omega_1', \quad \mu' = \beta\omega_2', \quad \nu' = \gamma\omega_3',$$

and (37) becomes

$$\alpha\dot{\omega}_1' - (\beta - \gamma)\,\omega_2'\,\omega_3' = 0, \text{ \&c.}$$

whence the motion of rotation is obtained by making the above mentioned quadric roll on the plane

$$\lambda x + \mu y + \nu z = \text{const.},$$

whose direction is fixed in space (since λ, μ, ν are constant), with an angular velocity proportional to the length OI of the radius vector drawn from the origin to the point of contact I.

The motion of translation is obtained by making the plane and quadric move through space with a velocity whose components are given by $\xi=0$, $\eta=0$, $\zeta=0$.

The theorems of the last two articles are taken from a paper by Prof. Lamb, *Proc. Lond. Math. Soc.* vol. VIII. p. 273.

EXAMPLES.

1. Apply Lagrange's equations to determine the equations of motion of an anchor ring; and thence obtain the theorem that the flux through the aperture relative to the ring, is the generalized velocity corresponding to the product of the circulation and density of the liquid.

2. If A and B be the forces required to act per unit of time, in order to generate unit velocity perpendicular and parallel respectively to the axis of an ellipsoid of revolution in an infinite liquid, and if G be the couple required to act per unit of time in order to generate unit angular velocity about an equatoreal axis, prove that the kinetic energy of the ellipsoid and the liquid is

$$\tfrac{1}{2}(Au^2+Av^2+Bw^2+G\omega_1^2+G\omega_2^2+C\omega_3^2)$$

with Euler's notation, C being the polar moment of inertia of the solid.

Express T in terms of Lagrange's coordinates $x, y, z, \theta, \phi, \psi$; and prove that if the axis of z be parallel to the impressed impulse F, then

$$\dot{x}=-F\left(\frac{1}{A}-\frac{1}{B}\right)\sin\theta\cos\theta\cos\psi,$$

$$\dot{y}=-F\left(\frac{1}{A}-\frac{1}{B}\right)\sin\theta\cos\theta\sin\psi,$$

$$\dot{z}=F\left(\frac{\sin^2\theta}{A}+\frac{\cos^2\theta}{B}\right),\ \dot{\phi}+\dot{\psi}\cos\theta=\omega_3,$$

$$G\dot{\psi}\sin^2\theta+C\omega_3\cos\theta=E \qquad \text{(a constant)},$$

$$G\dot{\theta}^2+G\dot{\psi}^2\sin^2\theta+C\omega_3^2+F^2\left(\frac{\sin^2\theta}{A}+\frac{\cos^2\theta}{B}\right)=2T.$$

3. In the midst of an infinite mass of liquid at rest, is a sphere of radius a, which is suddenly strained into a spheroid of small ellipticity. Find the kinetic energy due to the motion of the liquid contained between the given surface, and an imaginary concentric spherical surface of radius c; and show that if this imaginary surface were a real bounding surface which could not be deformed, the kinetic energy in this case would be to that in the former case in the ratio

$$c^5 (3a^5 + 2c^5) \ : \ 2 (c^5 - a^5)^2.$$

4. A pendulum with an elliptic cylindrical cavity filled with liquid, the generating lines of the cylinder being parallel to the axis of suspension, performs finite oscillations under the action of gravity. If l be the length of the equivalent pendulum, and l' the length when the liquid is solidified, prove that

$$l' - l = \frac{ma^2b^2}{h(M+m)(a^2+b^2)},$$

where M is the mass of the pendulum, m that of the liquid, h the distance of the centre of gravity of the whole mass from the axis of suspension, and a, b the semi-axes of the elliptic cavity.

5. Find the ratio of the kinetic energy of the infinite liquid surrounding an oblate spheroid, moving with given velocity in its equatoreal plane, to the kinetic energy of the spheroid; and denoting this ratio by P, prove that if the spheroid swing as the bob of a pendulum under gravity, the distance between the axis of the suspension and the axis of the spheroid being c, the length of the simple equivalent pendulum is

$$\frac{(1+P)\,c + 2a^2/5c}{1-\rho/\sigma},$$

where a is the equatoreal radius, σ and ρ the densities of the spheroid and liquid respectively.

6. A pendulum has a cavity excavated within it, and this cavity is filled with liquid. Prove that if any part of the liquid be solidified, the time of oscillation will be increased.

7. Prove that if a number of solids be moving freely under their mutual attractions in an unbounded liquid, the impulse of the motion remains constant.

8. The space between two infinitely long coaxial cylinders of radii a and b respectively, is filled with liquid of density ρ, and the inner cylinder is suddenly moved with velocity U perpendicular to the axis, the outer one being kept at rest. Show that the resultant impulsive pressure on a length l of the inner cylinder is

$$\pi\rho a^2 l U \frac{b^2+a^2}{b^2-a^2}.$$

9. An elliptic cylindrical shell, the mass of which may be neglected, is filled with water, and placed on a horizontal plane very nearly in the position of unstable equilibrium with its axis horizontal, and then let go. When it passes through the position of stable equilibrium, find the angular velocity of the cylinder, (i) when the horizontal plane is perfectly smooth, (ii) when it is perfectly rough; and prove that in these two cases, the squares of the angular velocities of the cylinder are in the ratio

$$(a^2-b^2)^2+4b^2(a^2+b^2) : (a^2-b^2)^2,$$

$2a$ and $2b$ being the axes of the cross section of the cylinder.

10. A solid ellipsoid of density σ is placed inside a fixed concentric, confocal, and similarly situated ellipsoidal shell, and the space between them is filled with liquid of density ρ. Supposing that the whole matter attracts according to the Newtonian law, and that $\sigma > \rho$, show that when the solid ellipsoid is slightly displaced parallel to its greatest axis, the time T of a small oscillation is given by

$$T^2\rho(\sigma-\rho)A/2\pi = \sigma+\rho-\frac{2\rho abc}{abc(2-A')-a'b'c'(2-A)},$$

where a, b, c and a', b', c' are the semi-axes of the outer and inner ellipsoids, and

$$A=\int_0^\infty \frac{abc\,d\lambda}{\{(a^2+\lambda)^3(b^2+\lambda)(c^2+\lambda)\}^{\frac{1}{2}}}.$$

11. The space between two coaxial cylinders is filled with liquid, and the outer is surrounded by liquid, extending to infinity, the whole being bounded by planes perpendicular to the axis. If the inner cylinder be suddenly moved with given velocity, prove that the velocity of the outer cylinder to that of the inner will be in the ratio

$$2b^2c^2\rho : \rho(a^2b^2-a^2c^2+b^4+b^2c^2)+\sigma(a^2-b^2)(b^2-c^2),$$

where a and b are the external and internal radii of the outer cylinder, σ its density, c the radius of the inner cylinder and ρ the density of the liquid.

12. The ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$, is filled with liquid originally at rest, and rotates uniformly about an axis through its centre of inertia: prove that the surfaces of equal pressure are given by the equation

$$(A, B, C, A', B', C')(x, y, z)^2 = \lambda,$$

where

$$A = \frac{(c^2 - a^2)(3c^2 + a^2)\,\omega_2^{\,2}}{(c^2 + a^2)^2} - \frac{(a^2 - b^2)(3b^2 + a^2)\,\omega_3^{\,2}}{(a^2 + b^2)^2},$$

$$A' = \frac{b^2c^2 + c^2a^2 + a^2b^2 - 3a^4}{(c^2 + a^2)(a^2 + b^2)}\,\omega_2\omega_3,$$

and $\omega_1, \omega_2, \omega_3$ are the component angular velocities of the ellipsoid.

13. In the last example prove that if the ellipsoid be set in rotation and then left to itself, the components of the velocity of the liquid relatively to the ellipsoid are

$$\dot{x} = \frac{2a^2\omega_3 y}{a^2 + b^2} - \frac{2a^2\omega_2 z}{a^2 + c^2},$$

$$\dot{y} = \frac{2b^2\omega_1 z}{b^2 + c^2} - \frac{2b^2\omega_3 x}{b^2 + a^2},$$

$$\dot{z} = \frac{2c^2\omega_2 x}{c^2 + a^2} - \frac{2c^2\omega_1 y}{c^2 + b^2},$$

and that if the ellipsoid revolves about a fixed axis after

$$\left\{\left(\frac{2bc\omega_1}{b^2 + c^2}\right)^2 + \left(\frac{2ca\omega_2}{c^2 + a^2}\right)^2 + \left(\frac{2ab\omega_3}{a^2 + b^2}\right)^2\right\}^{-\frac{1}{2}}$$

revolutions of the ellipsoid, every particle of liquid will be in the same position relatively to the ellipsoid.

14. A closed vessel filled with liquid of density ρ, is moved in any manner about a fixed point O. If at any time the liquid were removed, and a pressure proportional to the velocity potential were applied at every point of the surface, the resultant couple due to the pressure would be of magnitude G, and its direction in a line OQ. Show that the kinetic energy of the liquid was proportional to $\frac{1}{2}\rho\omega G\cos\theta$, where ω is the angular velocity of the surface, and θ the angle between the direction of ω and OQ.

15. A solid cylinder of radius a immersed in an infinite liquid, is attached to an axis about which it can turn, whose distance from the axis of the cylinder is c, and oscillates under the action

of gravity. Prove that the length of the simple equivalent pendulum is

$$\frac{\frac{1}{2}a^2 + c^2(1+\rho/\sigma)}{c(1-\rho/\sigma)},$$

σ and ρ being the densities of the cylinder and liquid.

16. A light cylindrical shell whose cross section is an ellipse filled with water is placed at rest on a smooth horizontal plane in its position of unstable equilibrium. If it is slightly disturbed, prove that it will pass through its position of stable equilibrium with angular velocity ω, given by the equation

$$\omega^2 = 8g\frac{a^2+b^2}{(a+b)^2(a-b)}.$$

17. A quantity of heavy heterogeneous liquid is placed inside an ellipsoid, which is then moved so that the density of the liquid is always the same function of the depth. Prove that a certain cone coaxial and concyclic with the reciprocal ellipsoid, moves so as always to have one of its generators vertical.

18. Liquid of density ρ is contained between two confocal elliptic cylinders and two planes perpendicular to their axes. The lengths of the semi-axes of the inner and outer cylinders are $c\cosh\alpha$, $c\sinh\alpha$, $c\cosh\beta$, $c\sinh\beta$ respectively. Prove that if the outer cylinder be made to rotate about its axis with angular velocity Ω, the inner cylinder will begin to rotate with angular velocity

$$\frac{\Omega\rho \operatorname{cosech} 2(\beta-\alpha)}{\rho\coth 2(\beta-\alpha)+\frac{1}{2}\sigma\sinh 4\alpha},$$

where σ is the density of the cylinder.

19. A circular cylinder of mass M, whose centre of inertia is at a distance c from its axis, is projected in an infinite liquid under the action of gravity. Prove that the centre of inertia of the cylinder and the displaced liquid will describe a parabola, while the cylinder oscillates like a pendulum of length

$$\{(M+M')k^2+M'c^2\}/2M'c,$$

where M' is the mass of the liquid displaced, and k is the radius of gyration of the cylinder about its axis.

20. The space between two coaxial similar and similarly situated elliptic cylinders is filled with liquid, and the cylinders are rotating with uniform angular velocity ω. Find what would be the new angular velocity if the liquid were suddenly solidified.

21. A hollow vessel of the form of an equilateral prism filled with liquid, is struck excentrically by a given blow in a plane perpendicular to the axis and bisecting three edges; find the initial motion of the vessel.

22. A cylinder whose cross section is an ellipse is moving in an infinite liquid. Prove that when there is circulation round the cylinder, its equations of motion are

$$\frac{d}{dt}(Pu\cos\theta - Qv\sin\theta + \kappa\rho y) = X,$$

$$\frac{d}{dt}(Pu\sin\theta + Qv\cos\theta - \kappa\rho x) = Y,$$

$$C\frac{d^2\theta}{dt^2} - (P-Q)\,uv = N,$$

where (x, y) are the coordinates of the centre of the cross section, X, Y the components of the impressed forces parallel to fixed axes, N is the impressed couple about the axis of the cylinder, u, v are the component velocities of the cylinder parallel to the major and minor axes of its cross section, and θ is the angle which the major axis makes with the axis of x.

23. Prove that helicoidal steady motion is always possible when a planetary ellipsoid is moving in an infinite liquid; but it is not possible in the case of an ovary ellipsoid, unless the ratio of the angular momentum of the ellipsoid about its polar axis, to its component velocity along this axis is greater than $2\sqrt{RA\,(1-R/P)}$; where R and P are the effective inertias of the ellipsoid about its polar axis, and an equatoreal axis and A is its effective moment of inertia about the latter axis.

24. A solid of revolution of mass M, is rotating in any manner about its centre of inertia, in an infinite liquid. Prove that if it is allowed to descend under the action of gravity, its vertical velocity at time t will be equal to

$$(M-M')\left(\frac{\sin^2\theta}{P} + \frac{\cos^2\theta}{R}\right)gt,$$

where M' is the mass of the liquid displaced; and θ is the inclination of the axis of the solid to the vertical at time t.

Obtain the differential equation for determining $d\theta/dt$.

CHAPTER X.

ON THE MOTION OF TWO CYLINDERS.

207. We have shown in Chapter V. that, when two cylinders are moving in a liquid of density ρ, the kinetic energy of the whole motion is

$$2T = (M+P)(u^2+v^2) + (M'+Q)(u'^2+v'^2) + 2L(uu'-vv'),$$

where M, M' are the masses of the cylinders; u, v, u', v' their component velocities perpendicular to and along the line joining their centres. The values of the coefficients are given[1] by equations (73) of § 123 or (74), (75) and (76) of § 124; and are functions of the distance between the cylinders alone.

208. We shall now apply these formulae to the consideration of the motion of a cylinder in a liquid bounded by a fixed plane, when there is no circulation[2].

When two equal cylinders are projected with equal velocities perpendicularly to the line joining their centres, it is clear that during the subsequent motion, the velocities of each cylinder perpendicular to this line will remain equal, and that their velocities parallel to this line will be equal and opposite. Hence the plane which is perpendicular to this line and bisects it will be fixed in space, and there will be no flux across it. One of the cylinders may therefore be removed, and the above mentioned plane substituted in its place; we shall thus obtain the motion of a cylinder in a liquid which is bounded by a rigid plane.

[1] See Errata.

[2] Hicks, "On the motion of two cylinders in a fluid," *Quart. Journ.*, vol. XVI. p. 193.

Let the axis of x lie in the plane, and be perpendicular to the axis of the cylinder; the kinetic energy of the liquid will be obtained by putting $\alpha=\beta$, $\theta_1=\theta_2=\sqrt{q}$; $u=u'$, $v=-v'$ in equations (74), (75) and (76) of § 124 and halving the result. Hence if σ be the density of the cylinder, and a its radius

$$2T=\{(P+L)+\pi a^2\sigma\}(u^2+v^2)$$
$$=R(u^2+v^2)\qquad\qquad(1),$$

where[1]
$$R=\pi a^2\rho\left\{1+2\Sigma_1^\infty\frac{(1-q)^2q^m}{(1-q^{m+1})^2}\right\}+\pi a^2\sigma.$$

If no external forces act upon the system, the energy, and also the momentum parallel to x, are constant; the latter condition gives

$$\frac{dT}{du}=\text{const.}=G,$$

or
$$Ru=G\qquad\qquad(2).$$

Since T and G are both constant, the equations of motion may now be written

$$\left.\begin{aligned}Ru&=G\\ R(u^2+v^2)&=2T\end{aligned}\right\}\qquad\qquad(3).$$

Differentiating with respect to t and remembering that R is a function of y alone, we obtain

$$\dot{v}+\frac{1}{2R}\frac{dR}{dy}(v^2-u^2)=0\qquad\qquad(4).$$

Now R is necessarily positive; also $y=a\cosh\alpha=\frac{1}{2}a(1+q)/q^{\frac{1}{2}}$, therefore R decreases as y increases; hence dR/dy is negative, and therefore $\dot{v}$ has always the same sign as v^2-u^2. Let U be the resultant velocity, ϕ the angle which its direction makes with the axis of y, then

$$\dot{v}=-\frac{U^2}{2R}\frac{dR}{dy}\cos 2\phi.$$

If therefore the direction of motion makes with the axis of y an angle lying between $\frac{1}{4}\pi$ and $\frac{3}{4}\pi$, the acceleration from the plane will be negative and the cylinder will be attracted towards the plane, but if this angle lies between 0 and $\frac{1}{4}\pi$ or $\frac{3}{4}\pi$ and π, the acceleration will be positive, and the cylinder will be repelled from the plane.

Also since $u=G/R$, and R decreases as y increases, u increases as the cylinder moves from the plane, and vice versâ.

[1] The value of $P+L$ in terms of elliptic functions will be given in the Appendix.

If we put
$$-\frac{U^2}{2R}\frac{dR}{dy}=f$$
the component accelerations are
$$\dot{u}=f\sin 2\phi, \qquad \dot{v}=f\cos 2\phi.$$

209. If the cylinder be initially in contact with the plane, and be projected perpendicularly from it, $u=0$, and
$$v^2=2T/R=v_0^2R_0/R,$$
where the suffixes denote the initial values of the quantities.

Since $q=0$ when $y=\infty$, the limiting value of R is $\pi a^2(\rho+\sigma)$. When $y=a$, $q=1$; in order to find the value of R_0, let $q=1-\lambda$, where λ is a small quantity which ultimately vanishes: then
$$R_0/\pi a^2=\rho\left\{1+2\left(\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots\right)\right\}+\sigma$$
$$=\rho\left(\tfrac{1}{3}\pi^2-1\right)+\sigma.$$
Whence the ratio of the ~~initial~~ terminal to the ~~terminal~~ initial velocity is
$$\sqrt{\frac{\frac{1}{3}\pi^2\rho+(\sigma-\rho)}{(\sigma+\rho)}}.$$

210. When the direction of projection is not perpendicular to the plane, the direction of the velocity at any subsequent time is given by the equation
$$\cot\phi=v/u=\pm\sqrt{Rp-1},$$
where $p=2T/G^2$, and the upper or lower sign must be taken according as the initial value of ϕ is $<$ or $>\frac{1}{2}\pi$. Let $\cot\phi$ be initially positive, so that the cylinder is projected from the plane, then since R diminishes to the limit $\pi a^2(\rho+\sigma)$ it follows that if $\pi a^2p(\rho+\sigma)<1$, there will be some point which is determined by the equation $Rp=1$, at which $\cot\phi=0$, and where the cylinder will consequently be moving parallel to the plane. During the subsequent motion $\cot\phi$ will be negative, and the cylinder will approach the plane and R will increase. The quantity $\sqrt{Rp-1}$ continually increases as R increases, and hence ϕ will increase from $\frac{1}{2}\pi$ and the cylinder will ultimately strike the plane. Hence the cylinder will or will not strike the plane according as $\pi a^2p(\rho+\sigma)<$ or >1.

If $\pi a^2p(\rho+\sigma)=1$, and α be the initial value of ϕ,
$$\cot\alpha=\sqrt{(R_0/\pi a^2-\rho-\sigma)/(\rho+\sigma)};$$

whence a cylinder projected at an angle $> \alpha$ will meet the plane at an angle

$$\tan^{-1} \sqrt{\{\pi a^2 p (\tfrac{1}{3}\pi^2 \rho + \sigma - \rho) - 1\}},$$

and a cylinder projected at an angle $< \alpha$ will move, when at an infinite distance from the plane in the direction

$$\cot^{-1} \sqrt{\{\pi a^2 p (\rho + \sigma) - 1\}}.$$

If the direction of projection is equal to α, the cylinder when at an infinite distance will move parallel to the plane.

211. Let one of the cylinders be fixed whilst the other moves independently.

Let (r, θ) be polar coordinates of the centre of the moving cylinder referred to the centre of the fixed cylinder as origin; if $R = P + M$; then

$$2T = R(\dot{r}^2 + r^2\dot{\theta}^2).$$

Since R is independent of θ, we must have

$$\frac{dT}{d\dot{\theta}} = \text{const.} = h,$$

or

$$Rr^2\dot{\theta} = h.$$

Also since

$$\frac{d}{dt}\frac{dT}{d\dot{r}} - \frac{dT}{dr} = 0,$$

we obtain

$$\ddot{r} - r\dot{\theta}^2 = -\frac{1}{2R}\frac{dR}{dr}(\dot{r}^2 - r^2\dot{\theta}^2).$$

Let U be the resultant velocity, ϕ the angle which its direction makes with the radius vector; the radial acceleration

$$f = -\frac{U^2}{2R}\frac{dR}{dr}\cos 2\phi.$$

Since R decreases as r increases dR/dr is negative; hence the cylinder will be repelled when ϕ lies between 0 and $\frac{1}{4}\pi$ or between $\frac{3}{4}\pi$ and π; and will be attracted if ϕ lies between $\frac{1}{4}\pi$ and $\frac{3}{4}\pi$.

212. If the cylinders be initially in contact, and one of them be projected with velocity V along the line joining their centres, then

$$\dot{r}^2 = 2T/R, \quad V^2 = 2T/R_0.$$

Therefore

$$\frac{\dot{r}^2}{V^2} = \frac{R_0}{R} = \frac{M + P_0}{M + P}.$$

If the cylinders are equal it can be shown in a similar manner as before, that

$$P_0 = \pi a^2 (\tfrac{1}{4}\pi^2 - 1)\rho,$$

or

$$P_\infty = \pi a^2 \rho,$$

whence

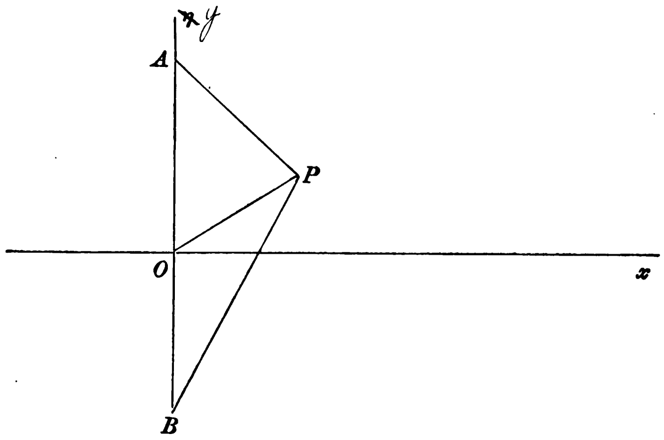
$$\frac{\dot{r}}{V} = \sqrt{\frac{\frac{1}{4}\pi^2\rho + \sigma - \rho}{\sigma + \rho}}.$$

Cyclic Motion.

213. Let us now consider the motion of two equal cylinders round which there is circulation in opposite directions, and which are initially projected with equal velocities parallel to Ox.

Let A and B be the common inverse points of the two cylinders, a the radius of either of them, u, v and $u, -v$ their velocities parallel and perpendicular to Ox, y the ordinate of the centre of the cylinder A; also let the circulation round A be in the contrary directions of the hands of a watch.

It is known from the theory of rectilinear vortices, which will be explained in Vol. II., that the cyclic motion is the same as



would be produced by two rectilinear vortices of circulations κ and $-\kappa$ situated at A and B, hence with the notation of § 178, the value of χ will be

$$\chi = -\frac{\kappa}{2\pi}\log\frac{AP}{BP} = \frac{\kappa\eta}{2\pi},$$

by § 121.

Also, if α be the value of η at the surface of the cylinder A, and $AB = 2c$,

$$a = c \operatorname{cosech} \alpha, \quad y = c \coth \alpha \quad \text{..................(5)},$$

and
$$\Sigma(\kappa\chi) = \kappa^2 a/\pi.$$

Since this kind of cyclic motion could be produced by applying a uniform impulsive pressure $\kappa\rho$ to every point of that portion of AB which lies between the cylinders, we must have $\mathfrak{P} = 0$. Let (r, θ) be the polar coordinates of P referred to O, then

$$\chi = -\frac{\kappa}{4\pi} \log \frac{r^2 + c^2 - 2rc \sin\theta}{r^2 + c^2 + 2rc \sin\theta} = \frac{\kappa c}{\pi r} \sin\theta + \&c.,$$

whence
$$\mathfrak{A} = 0, \quad \mathfrak{B} = -\kappa c/\pi.$$

Therefore
$$L = \mathfrak{T} + 2\kappa c\rho u - \kappa^2 \rho\alpha/2\pi + V$$

Also if M, M' be the masses per unit of length of either of the cylinders, and of the liquid displaced,

$$\mathfrak{T} = R(u^2 + v^2).$$

$$R = M' \left\{1 + 2(1-q)^2 \Sigma_1^\infty \frac{q^m}{(1-q^{m+1})^2}\right\} + M$$

where $q = \epsilon^{-2\alpha}$.

If we suppose the cylinder B to be replaced by the fixed plane Ox which forms the boundary of the liquid, the value of L must be halved, and the equations of motion of the cylinder A will be

$$\frac{d}{dt}\left(\tfrac{1}{2}\frac{d\mathfrak{T}}{du} + \kappa\rho c\right) = X \quad \text{....................(6)},$$

$$\tfrac{1}{2}\frac{d}{dt}\frac{d\mathfrak{T}}{dv} - \tfrac{1}{2}\frac{d\mathfrak{T}}{dy} - \kappa\rho u\frac{dc}{dy} + \frac{\kappa^2}{4\pi}\frac{d\alpha}{dy} = Y \quad \text{...........(7)}.$$

From (5) we obtain

$$c = \sqrt{y^2 - a^2} \text{ and } y = a\cosh\alpha,$$

therefore
$$\frac{dc}{dy} = \coth\alpha, \quad \frac{d\alpha}{dy} = \frac{1}{c},$$

whence (7) becomes

$$\tfrac{1}{2}\frac{d}{dt}\frac{d\mathfrak{T}}{dv} - \tfrac{1}{2}\frac{d\mathfrak{T}}{dy} - \kappa\rho u\coth\alpha + \frac{\kappa^2}{4\pi c} = Y \quad \text{.........(8)}.$$

Let us now suppose that gravity is the only force in action, and that the plane boundary Ox is horizontal, forming, so to speak, the bed of the ocean; (6) and (8) respectively become

$$\left.\begin{aligned} Ru + \kappa\rho c = \text{const.} = h \\ R\dot{v} + \tfrac{1}{2}(v^2 - u^2)\frac{dR}{dy} - \kappa\rho u\coth\alpha + \frac{\kappa^2\rho}{4\pi c} = -(M - M')g \end{aligned}\right\} \text{......(9)}.$$

These equations are satisfied by $v=0$, u and y constant, provided u satisfies the quadratic

$$pu^2 - \kappa\rho u \coth \alpha + \frac{\kappa^2\rho}{4\pi c} + (M - M')\, g = 0 \quad\text{.........(10)},$$

where $p = -\tfrac{1}{2}dR/dy$. The roots of this quadratic will be real

provided $$\kappa^2\rho^2 \coth^2\alpha > p\left\{\frac{\kappa^2\rho}{\pi c} + 4\,(M - M')\, g\right\} \quad\text{............(11)}.$$

Case (i). Since p is positive the roots will always be real if

$$M' > M$$

and $$\kappa^2\rho < \pi c\,(M' - M)\, g.$$

In this case the liquid is denser than the cylinder, and one of the roots of (10) will be positive and the other negative, and the positive root will be numerically greater than the negative root. Hence there will be two cases of steady motion, in one of which velocity of the cylinder will be in the *same* direction as that of the liquid, due to the circulation at points between the cylinder and plane; and in the other the velocity will be in the *opposite* direction; also the velocity in the former case will be greater than in the latter.

Case (ii). $M' > M, \quad \kappa^2\rho > 4\pi c\,(M' - M)\, g.$

In this case the roots of (10) will be both real and positive provided (11) is satisfied; hence the velocity in the two cases of steady motion will be in the *same* direction as that due to the circulation.

Case (iii). $M > M'$.

In this case the cylinder is denser than the liquid, and the roots of (10), if real, must be both positive, hence the two velocities must be in the *same* direction as that due to the circulation.

Case (iv). If either $g = 0$ or $M = M'$, (11) becomes

$$\pi\rho c \coth^2\alpha > p.$$

Here both roots of (10) are positive, and the two velocities must be in the same direction as that due to the circulation.

This case has been discussed by Mr W. M. Hicks[1].

[1] *Quart. Journ.* vol. XVII. p. 194.

CASE (v). Suppose that the cylinder is reduced to rest, and then let go. Since u and v are initially zero, the initial acceleration is

$$\dot{v} = -\frac{1}{4R\pi c}\{4\pi c(M - M')g + \kappa^2\rho\}\text{.........}(12).$$

Hence if the liquid is denser than the cylinder it is possible for the right-hand side to vanish; in which case the cylinder will remain in equilibrium under the combined action of gravity and the pressure due to the cyclic motion.

If the plane formed the upper boundary of the liquid the sign of g in these five cases would have to be reversed.

215. The results of the last two cases may be inferred from general reasoning.

We have shown in § 14, that the product of the velocity of a liquid and the cross section of a tube of flow, is constant throughout the length of the latter. Now in Case V. where the cylinder is at rest, the tubes of flow are circles, and those portions of them which lie between the cylinder and the plane will be more compressed than the portions which lie on the remote side of the cylinder; hence the velocity of the liquid at points between the cylinder and the plane will be on the whole greater than at points which lie on the opposite side of the cylinder, and consequently the pressure on the side of the cylinder nearest the plane will be less than that on the remote side, and therefore the cylinder will be attracted towards the plane. If the cylinder is *less dense* than the liquid, and the plane forms the lower boundary of the liquid, the effect of gravity will be to repel it from the plane, and hence there must be a certain position in which the two forces balance one another, and in which the cylinder will be in equilibrium. If on the other hand the plane forms the upper boundary of the liquid, there will be a position of equilibrium, provided the cylinder is *denser* than the liquid.

216. In Case IV. let the cylinder be moving with a small velocity u parallel to the plane, and in the same direction as that of the circulation between the cylinder and the plane. Let the cylinder be reduced to rest by impressing on the whole liquid a velocity u equal and opposite to that of the cylinder. At points between the cylinder and the plane, the reversed velocity u of the liquid and the velocity due to the circulation will be in opposite

directions, whilst at points on the other side of the cylinder they will be in the same direction. Also by § 14 each velocity will be on the whole greater at points between the cylinder and plane, than on the opposite side of the cylinder. Hence if u be small enough, the cylinder will be attracted towards the plane, and therefore if u increase from zero, a certain critical value u_1 will be reached, at which the cylinder is neither attracted nor repelled, but will be in equilibrium. In this case the resultant velocity at points between the cylinder and plane, will be in *the opposite direction* to that on the other side of the cylinder.

If u continue to increase, the cylinder will at first be repelled from the plane, but ultimately a second critical value u_2 will be reached, at which the resultant of u_2 and the velocity due to the circulation at points between the cylinder and the plane will on the average be equal to the same quantity on the opposite side of the cylinder, and there will be another position of equilibrium. In this case the resultant velocity of the liquid at points between the cylinder and the plane will be the *same direction* as that on the other side of the cylinder.

If u exceeds this second critical value the cylinder will thenceforth be attracted. The two critical values of u are evidently the roots of the quadratic obtained by putting $g = 0$ in (10).

EXAMPLES.

1. A cylinder of radius a is surrounded by a concentric cylinder of radius b, and the intervening space is filled with liquid. The inner cylinder is moved with velocity u and the outer with velocity v along the same straight line; prove that the velocity potential is

$$\phi = \frac{b^2v - a^2u}{b^2 - a^2} r \cos\theta + \frac{(v-u)a^2b^2\cos\theta}{(b^2-a^2)r}.$$

2. A long cylinder of given radius is immersed in a mass of liquid bounded by a very large cylindrical envelope. If the envelope be suddenly moved in a direction perpendicular to the cylinder with velocity V, the cylinder will begin to move with velocity $\frac{1}{2}V$, provided the density of the cylinder be three times that of the liquid.

3. Two infinite parallel cylinders in an infinite liquid are projected with given velocity; (i) in opposite directions along a line at right angles to their axes, (ii) in the same direction perpendicular to this line. Prove that they experience in the first instance a repulsion from one another, and in the second instance an attraction towards one another.

If their radii are indefinitely small in comparison with one another, prove that their motion is initially the same as that of two rectilinear vortices of equal and opposite strengths.

4. A solid cylinder with flat ends is fixed between two parallel planes, and a cylindrical shell of the same length can slide freely between the planes. If the space between the cylinder and shell is filled with liquid, and the shell is placed so as to be coaxial with the cylinder and then jerked in any direction with velocity V, prove that the resultant impulse on the cylinder is

$$2MVb^2(a^2-b^2),$$

where a and b are the radii of the cylinder and shell, and M is the mass of the liquid which the cylinder displaces.

5. The space between a moveable cylinder and a fixed *excentric* cylinder is filled with liquid. If the moveable cylinder be initially projected with given velocity, perpendicular to the line joining its centre with that of the fixed cylindrical boundary, determine its motion, (i) when there is no circulation, (ii) when there is circulation.

6. Examine the stability of the steady motion of a cylinder parallel to a fixed plane, discussed in § 214.

CHAPTER XI.

ON THE MOTION OF TWO SPHERES[1].

217. WHEN two spheres are in motion in an infinite liquid, the velocity of each sphere may be resolved into three components u_1, v_1, w_1; u_2, v_2, w_2, where u_1, u_2 are the component velocities of the spheres along the line joining their centres; and v_1, w_1; v_2, w_2 are the component velocities parallel to two straight lines at right angles to one another, which are perpendicular to the line joining the centres of the two spheres. It would therefore at first sight appear, that the kinetic energy of the liquid must contain twenty-one terms, but it can easily be shown that twelve of these terms must vanish. For let us suppose that v_1, w_1, u_2, v_2 are each zero, and consider the term involving $u_1 w_2$. The kinetic energy on

[1] The present chapter has been taken from the following papers by Mr Hicks:

"On the Motion of Two Spheres in a Fluid," *Phil. Trans.* 1880, p. 455.

"On the Problem of Two Pulsating Spheres in a Fluid," *Proc. Camb. Phil. Soc.* vol. III. p. 277, and vol. IV. p. 29;

and a paper by the author,

"On the Motion of Two Spheres in a Liquid and allied Problems," *Proc. Lond. Math. Soc.* vol. XVIII. p. 369.

References may also be made to the following papers:

Stokes. "On some Cases of Fluid Motion," *Trans. Camb. Phil. Soc.* vol. VIII. p. 105.

Bjerknes. *Forhand. Skand. Naturfors*, Christiania 1868, and *Forhand. Vidensk.*, Christiania 1871 and 1875.

G. Forbes. "Hydrodynamic analogies to Electricity and Magnetism," *Nature*, vol. XXIV. p. 360.

Bertin. "Phénomènes Hydrodynamiques inversement analogues à ceux de l'Électricité et du Magnétisme," *Ann. de Chimie et de Phys.* (5) XXV. p. 257, 1882.

Pearson. "On the Motion of Spherical and Ellipsoidal bodies in Fluid Media," *Quart. Journ.* vol. XX. p. 60.

Herman. "On the Motion of Two Spheres in a Fluid and allied Problems," *Quart. Journ.* vol. XXII. p. 204.

account of the symmetry of the motion, must clearly be unaltered if the direction of w_2 be reversed, and this requires that the coefficient of u_1w_2 should be zero. By similar reasoning it can be shown that all the other coefficients must vanish, except those of u_1^2, u_2^2, v_1^2, v_2^2, w_1^2, w_2^2, u_1u_2, v_1v_2, w_1w_2; and also that the coefficients of v_1^2, v_2^2, v_1v_2 must be respectively equal to those of w_1^2, w_2^2, w_1w_2.

Hence the kinetic energy of the system may be written

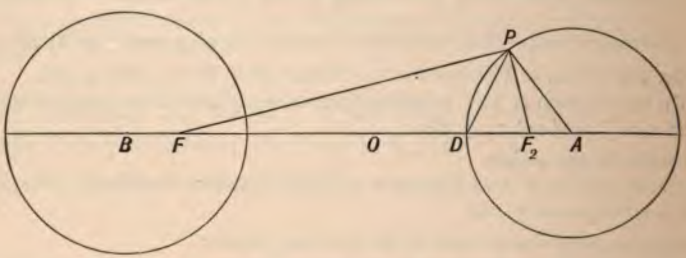
$$T = \tfrac{1}{2}(Au_1^2 - 2Bu_1u_2 + Cu_2^2) + \tfrac{1}{2}A'(v_1^2 + w_1^2) \\ + B'(v_1v_2 + w_1w_2) + \tfrac{1}{2}C'(v_2^2 + w_2^2),$$

where the six coefficients are functions of the distance between the centres of the two spheres and their radii.

The values of A, B and C must be determined by supposing that the motion of the spheres is along the line joining their centres, and those of A', B', C' by supposing that the motion is perpendicular to this line.

Motion along the Line of Centres.

218. Let A and B be the centres of the spheres, a and b their radii, c the distance between their centres.



Let ϕ_1 be the velocity potential when A is moving with velocity u_1 along BA and B is at rest; ϕ_2 the velocity potential when B is moving with velocity u_2 along the same direction and A is at rest. By § 162 the velocity potential of the whole motion is $\phi_1 + \phi_2$, and the kinetic energy of the liquid is

$$\left.\begin{aligned} \mathfrak{T} &= -\tfrac{1}{2}\rho\iint\phi_1\frac{d\phi_1}{dn}dS_1 - \rho\iint\phi_2\frac{d\phi_1}{dn}dS_1 - \tfrac{1}{2}\rho\iint\phi_2\frac{d\phi_2}{dn}dS_2 \\ &= T_{11} + 2T_{12} + T_{22} \end{aligned}\right\} \ldots(1).$$

In order to find the value of $\mathfrak{T}$ we shall employ the method of images.

If B were absent, the velocity potential due to the motion of A, would be the same as that of a positive doublet[1] at A of strength $\frac{1}{2}u_1a^3$, whose axis coincides with BA. By § 53 the image of this in B, is a negative doublet situated at the inverse point F, where $BF.BA = b^2$, and whose strength is $-u_1a^3b^3/2c^3$. This latter doublet will have an image in A, and so on ad infinitum. Hence the kinetic energy of the liquid due to the motion of the sphere A, will be the same as that due to two infinite systems of doublets, both of which lie respectively within each sphere.

219. Let ρ_n be the distance of the nth image in A from A, μ_n its strength; and let σ_n be the distance of the nth image in B from A, ν_n its strength. The part of T_{11} due to μ_n will be

$$\pi\rho a^2u_1\int_0^\pi \frac{\mu_n(a\cos\theta+\rho_n)\sin\theta\cos\theta\,d\theta}{(a^2+\rho_n^2+2a\rho_n\cos\theta)^{\frac{3}{2}}}$$

$$= \pi\rho a^2u_1\mu_n\int_{-1}^1 \frac{(\rho_n+ax)\,x\,dx}{(a^2+\rho_n^2+2a\rho_n x)^{\frac{3}{2}}}.$$

But

$$\int_{-1}^1 \frac{(r+ax)\,x\,dx}{(a^2+r^2+2arx)^{\frac{3}{2}}} = -\frac{d}{dr}\int_{-1}^1 \frac{x\,dx}{(a^2+r^2+2arx)^{\frac{1}{2}}}$$

$$= -\frac{d}{dr}\frac{1}{3a^2r^2}\{(a+r)(ar-a^2-r^2) \pm (a-r)(a^2+r^2+ar)\}.$$

When $r=\rho_n<a$, the integral is equal to

$$\frac{d}{dr}\frac{2r}{3a^2} = \frac{2}{3a^2} \quad\text{.........................(2).}$$

But when $r=\sigma_n>a$, it equals

$$\frac{d}{dr}\frac{2a}{3r^2} = -\frac{4a}{3r^3} \quad\text{.........................(3).}$$

Therefore $T_{11} = \frac{2}{3}\pi\rho u_1\Sigma_0^\infty\,\mu_n - \frac{8}{3}\pi\rho a^3u_1\Sigma_1^\infty\,\nu_n\sigma_n^{-3}$.

Now $\mu_0 = \frac{1}{2}a^3u_1,\quad \mu_n = -a^3\nu_n\sigma^{-3}$.

[1] A doublet is considered positive when its source end is at the positive extremity of its axis. If m be its strength, its velocity potential is $-mr^{-2}\cos\theta$.

Hence if M_1 be the mass of the liquid displaced by the sphere A,

$$T_{11} = \tfrac{1}{4} M_1 u_1^2 \left(1 + 3\Sigma_1^\infty \frac{\mu_n}{\mu_0}\right) \text{.................. (4).}$$

This is the kinetic energy due to the surface integral of A's motion over itself.

Again,

$$\nu_n = -b^3\mu_{n-1}/(c-\rho_{n-1})^3,\ \rho_n = a^2/\sigma_n,\ c - \sigma_n = b^2/(c-\rho_{n-1}) \text{...(5),}$$

whence $$\mu_n = \frac{a^3 b^3 \mu_{n-1}}{\sigma_n^{\,3}(c-\rho_{n-1})^3} = \frac{b^3 \rho_n^{\,3} \mu_{n-1}}{a^3(c-\rho_{n-1})^3}$$

$$= \left(\frac{a}{b}\right)^{3n} \left\{\frac{\rho_n \rho_{n-1} \cdots \rho_1}{(c-\rho_{n-1})(c-\rho_{n-2})\cdots(c-\rho_1)c}\right\}^3 \mu_0 \text{ ......(6).}$$

Eliminating σ_n from (5) we obtain

$$c\rho_n\rho_{n-1} - (c^2 - b^2)\rho_n - a^2\rho_{n-1} + a^2 c = 0 \text{............(7).}$$

220. The formulae of the preceding section enable us to obtain an approximate value of T_{11} as far as c^{-12} without much difficulty, but in order to obtain the complete solution we must solve (7). To do this, put $\rho_n = u_n + x$, and choose x so as to make the constant term vanish, and we obtain

$$cx^2 - (a^2 + c^2 - b^2)x + a^2 c = 0 \text{ ....(8).}$$

Let F, F_2 be the common inverse points of the two spheres, O the middle point of FF_2; also let $FF_2 = 2\lambda$, $OA = r_1$, $OB = r_2$, then

$$\left.\begin{array}{ll} & r_1^{\,2} - \lambda^2 = AF_2 \,.\, AF = a^2 \\ & r_2^{\,2} - \lambda^2 = b^2, \\ \text{therefore} & r_1^{\,2} - r_2^{\,2} = a^2 - b^2; \\ \text{also} & r_1 + r_2 = c, \\ \text{therefore} & r_1 = (a^2 + c^2 - b^2)/2c \end{array}\right\} \text{...............(9).}$$

Let P be any point on the sphere A, and let the constant ratio F_2P/FP be denoted by q_1, and let q_2 be the similar constant for the sphere B. Then since the triangles PF_2A and FPA are similar,

$$q_1 = F_2A/a = (r_1 + \lambda)/a = a/(r_1 - \lambda),$$
$$q_2 = b/(r_2 + \lambda) = (r_2 - \lambda)/b,$$

and (8) becomes

$$x^2 - 2r_1 x + a^2 = 0,$$

the roots of which are $x_1 = r_1 + \lambda$, $x_2 = r_1 - \lambda$. Putting $\rho_n = u_n + x_1$, equation (7) may now be written

$$u_n u_{n-1} - (x_2 - a^2/c)\, u_n + (x_1 - a^2/c)\, u_{n-1} = 0.$$

Now $a^2 = x_1 x_2$, whence, writing v_n^{-1} for u_n, we obtain

$$v_n - \frac{x_2(c - x_1)}{x_1(c - x_2)} v_{n-1} = -\frac{c}{x_1(c - x_2)}.$$

In this equation

$$\frac{c - x_1}{c - x_2} = \frac{r_2 - \lambda}{r_2 + \lambda} = q_2^2,$$

$$\frac{x_1}{x_2} = \frac{r_1 + \lambda}{r_1 - \lambda} = q_1^2,$$

$$\frac{c}{x_1(c - x_2)} = \frac{c}{(r_1 + \lambda)(r_2 + \lambda)}.$$

Whence putting $q = q_2/q_1$, we obtain

$$v_{n+1} - q^2 v_n = -\frac{c}{(r_1 + \lambda)(r_2 + \lambda)},$$

the solution of which is

$$v_n = Eq^{2n} - \tfrac{1}{2}\lambda^{-1},$$

hence $$\rho_n = aq_1 + (Eq^{2n} - \tfrac{1}{2}\lambda^{-1})^{-1}.$$

But $\rho = 0$ when $n = 0$, therefore

$$E = \frac{1}{2\lambda} - \frac{1}{r_1 + \lambda} = \frac{r_1 - \lambda}{2\lambda(r_1 + \lambda)} = \frac{1}{2\lambda q_1^2},$$

therefore $$\rho_n = aq_1 - \frac{2\lambda}{1 - q^{2n} q_1^{-2}}$$

$$= (r_1 - \lambda)\frac{1 - q^{2n}}{1 - q^{2n} q_1^{-2}}.$$

Also $$c - \rho_n = r_1 + r_2 - r_1 - \lambda + 2\lambda/(1 - q^{2n} q^{-2})$$

$$= (r_2 + \lambda)\frac{1 - q^{2n+2}}{1 - q^{2n} q_1^{-2}},$$

therefore $$\frac{b\rho_n}{a(c - \rho_n)} = \frac{q(1 - q^{2n-2} q_1^{-2})}{1 - q^{2n} q_1^{-2}} = \frac{qp_{n-1}}{p_n} \text{ (say)};$$

therefore $$\mu_n = \left\{q^n \frac{p_{n-1} p_{n-2} \cdots\cdots p_0}{p_n p_{n-1} \cdots\cdots p_1}\right\}^3 \mu_0$$

$$= \left\{\frac{(1 - q_1^{-2}) q^n}{1 - q^{2n} q_1^{-2}}\right\}^3 \mu_0.$$

If therefore we put

$$Q(q_1^{-1}, q) = (1 - q_1^{-2})^3 \Sigma_1^\infty \left(\frac{q^n}{1 - q^{2n} q_1^{-2}}\right)^3 \quad \text{.........(10)},$$

we obtain $$T_{11} = \tfrac{1}{4} M_1 u_1^2 \{1 + 3Q(q_1^{-1}, q)\} \quad \text{...............(11)}.$$

Similarly if the sphere B were moving with velocity u_2 along BA whilst A is fixed, it can be shown that

$$T_{22} = \tfrac{1}{4} M_2 u_2^2 \{1 + 3Q(q_2, q)\} \quad \text{...............(12)}.$$

221. We must now calculate the quantity T_{12} which is the surface integral of B's motion taken over A, and which by Green's theorem is equal to the surface integral of A's motion taken over B. We thus obtain,

$$T_{12} = -\tfrac{1}{2}\rho \iint \phi_2 \frac{d\phi_1}{dn} dS_1 = -\pi\rho a^2 u_1 \int_0^\pi \phi_2 \sin\theta \cos\theta d\theta.$$

Let ρ_n' denote the distance from A of the nth image of B in A, μ_n' its strength; also let σ_n' denote the distance of the nth image in B from A, ν_n' its strength; then remembering that the original doublet is in B, we obtain

$$\mu_n' = -(a/\sigma'_{n-1})^3 \nu'_{n-1}, \; \mu_1' = -(a/c)^3 \nu_0' = -\tfrac{1}{2} a^3 b^3 c^{-3} u_2 \quad \text{...(13)}.$$

Hence

$$T_{12} = -\tfrac{4}{3}\pi\rho a^3 u_1 \Sigma_0^\infty \left(\frac{\nu_n'}{\sigma_n'^3}\right) + \tfrac{2}{3}\pi\rho u_1 \Sigma_1^\infty \mu_n'$$

$$= 2\pi\rho u_1 \Sigma_1^\infty \mu_n' = -\pi\rho u_1 u_2 \frac{a^3 b^3}{c^3} \Sigma_1^\infty \left(\frac{\mu_n'}{\mu_1'}\right) \quad \text{......(14)}.$$

Also

$$\rho_1' = a^2/c, \quad \sigma_1' = b^2/(c - \rho_1'), \quad \rho_2' = a^2/(c - \sigma_1') \quad \text{......(15)},$$

whence, proceeding as before, it will be found that

$$\mu_n' = \left\{\frac{b\rho_n'}{a(c - \rho'_{n-1})}\right\}^3 \mu'_{n-1}$$

$$= \left(\frac{b}{a}\right)^{3n-3} \left\{\frac{\rho_n' \rho'_{n-1} \cdots\cdots \rho_2'}{(c - \rho'_{n-1}) \cdots\cdots (c - \rho_1')}\right\}^3 \mu_1' \quad \text{.........(16)},$$

and it can be shown as before that

$$\rho_n' = aq_1 + (Eq^{2n} - \tfrac{1}{2}\lambda^{-1})^{-1}.$$

Whence, determining E by the condition that $\rho_1' = a^2/c$, we shall find

$$\rho_n' = aq_1 - 2\lambda(1-q^{2n})^{-1} = (r_1-\lambda)\frac{1-q^{2n}q_1^2}{1-q^{2n}},$$

$$c-\rho'_{n-1} = (r_2+\lambda)\frac{1-q^{2n}q_1^2}{1-q^{2n-2}},$$

$$\frac{b\rho_n'}{a(c-\rho'_{n-1})} = \frac{q(1-q^{2n-2})}{1-q^{2n}}.$$

Therefore $$\mu_n' = \left\{\frac{(1-q^2)q^{n-1}}{1-q^{2n}}\right\}^3\mu_1'.$$

Now from (9) we obtain

$$q^2 = \frac{(r_2-\lambda)(r_1-\lambda)}{(r_2+\lambda)(r_1+\lambda)},$$

therefore $$1-q^2 = \frac{2\lambda c}{(r_2+\lambda)(r_1+\lambda)} = \frac{2\lambda cq}{ab}.$$

If therefore we put

$$Q_1(q) = \Sigma_1^\infty\left(\frac{2\lambda q^n}{1-q^{2n}}\right)^3 \quad\text{......................(17)},$$

we obtain

$$T_{12} = -\pi\rho u_1u_2Q_1(q) \quad\text{.....................(18)}.$$

Hence, if m_1, m_2 be the masses of the two spheres, the kinetic energy of the whole motion when the spheres are moving along the line joining their centres is

$$T = \tfrac{1}{2}(Au_1^2 - 2Bu_1u_2 + Cu_2^2) \quad\text{...............(19)},$$

where
$$\left.\begin{aligned} A &= m_1 + \tfrac{1}{2}M_1\{1+3Q(q_1^{-1}, q)\} \\ C &= m_2 + \tfrac{1}{2}M_2\{1+3Q(q_2, q)\} \\ B &= 2\pi\rho u_1u_2Q_1(q) \end{aligned}\right\} \quad\text{............(20)}.$$

The three coefficients A, B, C can be shown to diminish as the distance between the spheres increases; for when c and therefore λ is large,

$$q_1 = (r_1+\lambda)/a = 2\lambda/a,$$
$$q_2 = b/(r_2+\lambda) = b/2\lambda,$$
$$q = ab/4\lambda^2,$$

ultimately, and therefore A, B, and C diminish as c increases. Also, since T is essentially a positive quantity, $AC > B^2$.

222. The general formulae (20) are too complicated to be of much use, we shall therefore obtain approximate values of A, B and C as far as c^{-12}.

From (5) and (6) we obtain

$$\rho_1 = a^2c/(c^2 - b^2),$$

whence
$$\frac{\mu_1}{\mu_0} = \left(\frac{b\rho_1}{ac}\right)^3 = \left(\frac{ab}{c^2 - b^2}\right)^3 \quad \text{..................(21)};$$

also from (6)
$$\frac{\mu_2}{\mu_0} = \left\{\frac{b^2\rho_2\rho_1}{a^2c(c - \rho_1)}\right\}^3.$$

From (7)
$$\rho_2(c\rho_1 - c^2 + b^2) = -a^2(c - \rho_1),$$

therefore
$$\frac{\rho_2}{c - \rho_1} = \frac{a^2}{c^2 - b^2 - c\rho_1} = \frac{a^2(c^2 - b^2)}{(c^2 - b^2)^2 - a^2c^2},$$

whence
$$\frac{\mu_2}{\mu_0} = \left\{\frac{a^2b^2}{(c^2 - b^2)^2 - a^2c^2}\right\}^3 \quad \text{.....................(22)}.$$

The last expression varies as c^{-12}, whence expanding the values of μ_1/μ_0, μ_2/μ_0 in powers of c^{-1}, and neglecting higher powers than c^{-12}, we obtain

$$A = m_1 + \tfrac{1}{2}M_1\left\{1 + \frac{3a^3b^3}{c^6}\left(1 + \frac{3b^2}{c^2} + \frac{6b^4}{c^4} + \frac{11b^6}{c^6}\right)\right\},$$

and the value of C can be obtained by interchanging a and b.

To determine B to the same order, we obtain from (16)

$$\frac{\mu_2'}{\mu_1'} = \left\{\frac{b\rho_2'}{a(c - \rho_1')}\right\}^3 = \frac{a^3b^3}{(c^2 - a^2 - b^2)^3} \quad \text{...........(23)},$$

whence
$$B = \frac{2\pi\rho a^3b^3}{c^3}\left\{1 + \frac{a^3b^3}{c^6} + \frac{3a^3b^3(a^2 + b^2)}{c^8}\right\}.$$

Collecting our results, the values of A, B and C as far as c^{-12} are

$$\left.\begin{aligned}
A &= m_1 + \tfrac{1}{2}M_1\left\{1 + \frac{3a^3b^3}{c^6}\left(1 + \frac{3b^2}{c^2} + \frac{6b^4}{c^4} + \frac{11b^6}{c^6}\right)\right\}\\
C &= m_2 + \tfrac{1}{2}M_2\left\{1 + \frac{3a^3b^3}{c^6}\left(1 + \frac{3a^2}{c^2} + \frac{6a^4}{c^4} + \frac{11a^6}{c^6}\right)\right\}\\
B &= \frac{2\pi\rho a^3b^3}{c^3}\left\{1 + \frac{a^3b^3}{c^6} + \frac{3a^3b^3(a^2 + b^2)}{c^8}\right\}
\end{aligned}\right\} \text{...(24)}.$$

Motion perpendicular to the Line of Centres.

223. When the spheres are moving perpendicularly to the line joining their centres, the kinetic energy may be determined by the method of images without much difficulty, provided we neglect powers of c^{-1} higher than the eighth; but if it is desired to carry the approximation to a higher degree, the successive images become exceedingly complicated, and it is better to employ a different method, which will be explained later on.

224. Let v_1, v_2 be the velocities of A and B perpendicular to AB. If B were absent the velocity potential due to A's motion would be the same as that due to a positive doublet at A, of strength $\frac{1}{2}v_1a^3$, whose axis is perpendicular to AB. By § 54 the image of this in B, is a positive doublet of strength $\frac{1}{2}v_1a^3b^3c^{-3}$ situated at the inverse point F, together with a negative line doublet extending from F to B, whose strength at any point P is $-\frac{1}{2}v_1a^3BP/bc$ per unit of length. Hence the successive images consist of a series of single doublets and line doublets, and evidently become exceedingly complicated.

Let χ be the angle which any plane through AB makes with the direction of motion of the spheres, r the distance of any doublet element from A, μ its strength. The kinetic energy will be given by an expression of the same form as (1), whence the part of T_{11} depending on μ will be

$$\frac{1}{2}\int_0^\pi\int_0^{2\pi}\frac{v_1\rho a^3\mu\sin^3\theta\cos^2\chi\,d\theta\,d\chi}{(r^2+a^2+2ar\cos\theta)^{\frac{3}{2}}}$$
$$=\frac{1}{2}\pi\rho a^3v_1\mu\int_0^\pi\frac{\sin^3\theta\,d\theta}{(r^2+a^2+2ar\cos\theta)^{\frac{3}{2}}}.$$

The value of this integral is

$$\frac{r^2+a^2}{a^3r^3}\{r+a\mp(r-a)\}-\frac{1}{3a^3r^3}\{(r+a)^3\mp(r-a)^3\},$$

in which the upper or lower sign is to be taken according as $r >$ or $< a$. Hence the value of the integral is

$$\tfrac{4}{3}a^{-3},\quad a>r;\ \text{ and }\ \tfrac{4}{3}r^{-3},\quad r>a$$

and therefore the part of T_{11} depending on μ is $\frac{2}{3}\pi\rho\mu v_1$ or $\frac{2}{3}\pi\rho\mu v_1a^3/r^3$, according as $r <$ or $> a$.

Let ν and σ be the strengths and distances from A, of the doublets within B due to A's motion, and μ the strengths of the doublets within A. Then

$$T_{11} = \tfrac{2}{3}\pi\rho v_1 \Sigma\,(\mu + \nu a^3\sigma^{-3})$$
$$= \tfrac{1}{2}M_1 v_1 \Sigma\left(\frac{\mu}{a^3} + \frac{\nu}{\sigma^3}\right).$$

Now every ν produces in A an image consisting of a doublet of strength $\nu a^3/\sigma^3$ at a distance a^2/σ from the centre of A, together with a negative line doublet extending from the doublet image to the centre of A, and whose line strength at a point whose distance from A is x, is $-\nu x/a\sigma$. Hence the whole amount of the image is

$$\nu\left(\frac{a}{\sigma}\right)^3 - \frac{\nu}{2a\sigma}\left(\frac{a^2}{\sigma}\right)^2 = \tfrac{1}{2}\nu\left(\frac{a}{\sigma}\right)^3.$$

Also every μ except μ_0 forms part of an image of some particular ν, hence

$$\Sigma\,\frac{\nu}{\sigma^3} = 2\Sigma\,\frac{\mu}{a^3} - \frac{2\mu_0}{a^3}.$$

Therefore

$$T_{11} = \frac{M_1 v_1}{2a^3}\Sigma_1^\infty\,(\mu_0 + 3\mu)$$
$$= \tfrac{1}{4}M_1 v_1^2\left\{1 + 3\Sigma_1^\infty\left(\frac{\mu_n}{\mu_0}\right)\right\} \quad\text{..............(25).}$$

225. In order to find the term involving $v_1 v_2$, we must find the portion of the kinetic energy due to B's motion over A and double the result.

Since the original doublet is in B, every ν except ν_0 forms part of an image of some μ, whence if the accented letters refer to the images of B's motion

$$\Sigma_1^\infty\,\frac{\nu'}{\sigma^3} = 2\Sigma_1^\infty\,\frac{\mu'}{a^3},$$

hence

$$T_{12} = \tfrac{2}{3}\pi\rho v_1 \Sigma\left(\frac{\nu' a^3}{\sigma^3} + \mu'\right)$$
$$= 2\pi\rho v_1 \Sigma_1^\infty\,\mu'$$
$$= \pi\rho v_1 v_2 b^3\,\Sigma_1^\infty\,\frac{\mu'}{\nu_0'} \quad\text{........................(26),}$$

and we therefore obtain

$$T = \tfrac{1}{2}(A'v_1^2 + 2B'v_1v_2 + C'v_2^2),$$

where

$$\left.\begin{aligned} A' &= m_1 + \tfrac{1}{2}M_1\left\{1 + 3\Sigma_1^\infty\left(\frac{\mu_n}{\mu_0}\right)\right\} \\ C' &= m_2 + \tfrac{1}{2}M_2\left\{1 + 3\Sigma_1^\infty\left(\frac{\nu_n}{\nu_0}\right)\right\} \\ B' &= 2\pi\rho b^3\,\Sigma_1^\infty\left(\frac{\mu_n'}{\nu_0'}\right) \end{aligned}\right\} \quad \text{............(27).}$$

226. We shall now calculate the values of the coefficients when all the images are omitted except μ_1, ν_1, μ_1'.

The image of A in B consists of a doublet of strength $\mu_0 b^3/c^3$ at F, together with a negative line doublet from F to B of strength $-\mu_0 x/bc$ per unit of length; also $BF = b^2/c$, $AF = (c^2 - b^2)/c$.

The image in A of the doublet at F is a doublet at a point F' whose strength is

$$\frac{\mu_0 a^3 b^3}{c^3 AF^3} = \frac{\mu_0 a^3 b^3}{(c^2 - b^2)^3} \text{ ....(28),}$$

where $AF' = a^2c/(c^2 - b^2)$, together with a negative line doublet from F' to A whose whole amount is

$$-\frac{\mu_0 b^3}{c^3}\int_0^{AF'} \frac{y\,dy}{aAF} = -\frac{\mu_0 a^3 b^3}{2(c^2 - b^2)^3} \text{ ....(29).}$$

In order to find the whole amount of the image of the line doublet between B and F, let P be any point in BF, Q a point on AF' such that $AP\,.\,AQ = a^2$; also let $BP = x$, $AQ = y$; then $y(c - x) = a^2$. The doublet element $-\mu_0 x\,dx/bc$ at P, produces a doublet element $-\mu_0 x a^3 dx/bc\,(c - x)^3$ at Q, together with a line doublet from Q to A whose whole amount is

$$\frac{\mu_0 x\,dx}{bc}\int_0^{\frac{a^2}{c-x}} \frac{y\,dy}{a(c - x)} = \frac{\mu_0 a^3 x\,dx}{2bc\,(c - x)^3}.$$

Therefore the whole amount of the line doublet is

$$-\frac{\mu_0 a^3}{2bc}\int_0^{b^2/c} \frac{x\,dx}{(c - x)^3} = -\frac{\mu_0 a^3 b^3}{4c^2(c^2 - b^2)^2} \text{ ....(30),}$$

adding (28), (29) and (30) we obtain

$$\frac{\mu_1}{\mu_0} = \tfrac{1}{2}\left\{\left(\frac{ab}{c^2 - b^2}\right)^3 - \frac{a^3 b^3}{2c^2(c^2 - b^2)^2}\right\} \text{ ....(31).}$$

Again $$\mu_1' = \frac{\nu_0' a^3}{c^3} - \nu_0'\int_0^{a^2/c} \frac{x\,dx}{ac} = \frac{\nu_0' a^3}{2c^3} \text{ ....(32),}$$

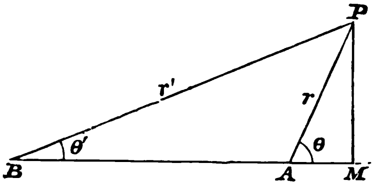
whence substituting from (31) and (32) in (27), we obtain

$$\left.\begin{aligned} A' &= m_1 + \tfrac{1}{2}M_1\left\{1 + \tfrac{3}{2}\left(\frac{ab}{c^2-b^2}\right)^3 - \frac{3a^3b^3}{4c^2(c^2-b^2)^2}\right\} \\ C' &= m_2 + \tfrac{1}{2}M_2\left\{1 + \tfrac{3}{2}\left(\frac{ab}{c^2-a^2}\right)^2 - \frac{3a^3b^3}{4c^2(c^2-a^2)^2}\right\} \\ B' &= \frac{\pi\rho a^3b^3}{c^3} \end{aligned}\right\} \ldots(33).$$

The second ratio μ_2/μ_0 is of the order c^{-12}, and the next term in B' is of the order c^{-9}. Hence (33) gives the correct values of A' and C' as far as c^{-10}, and the expression for the kinetic energy derived from (33) is correct as far as c^{-8}.

227. We shall now explain a different method for obtaining approximate values of the coefficients[1]. The approximation is carried as far as c^{-12}, but it could without much additional labour be carried to a higher order if desired.

It will first be necessary to establish the following proposition.



In the figure, let $PM = \varpi$, $AM = z$, $BM = z'$, $AB = c$, $\cos\theta = \mu$, $\cos\theta' = \mu'$; also let $P_n^m(\mu)$ be an associated function of degree n and order m, whose origin is A, and axis is AM; and let $P'^m_n(\mu')$ denote a similar function having the same axis and whose origin is B. Then we shall prove that, when $r < c$,

$$\frac{P'^m_n}{r'^{n+1}} = \frac{r^m}{(n-m)!\,c^{n+m+1}}\left[\frac{(n+m)!}{2m!}P_m^m - \frac{(n+m+1)!}{(2m+1)!}\frac{r}{c}P_{m+1}^m + \ldots + \frac{(-)^s(n+m+s)!}{(2m+s)!}\left(\frac{r}{c}\right)^s P_{m+s}^m + \ldots\right] \ldots\ldots\ldots(34),$$

and when $r' < c$,

$$\frac{(-)^{n-m}P_n^m}{r^{n+1}} = \frac{r'^m}{(n-m)!\,c^{n+m+1}}\left[\frac{(n+m)!}{2m!}P'^m_m + \frac{(n+m+1)!}{(2m+1)!}\frac{r'}{c}P'^m_{m+1} + \frac{(n+m+s)!}{(2m+s)!}\left(\frac{r'}{c}\right)^s P'^m_{m+s} + \ldots\right] \ldots\ldots\ldots(35).$$

[1] *Proc. Lond. Math. Soc.* vol. XVIII. p. 371.

It is known that P_n^m can be expressed in either of the forms[1]

$$M(1-\mu^2)^{\frac{1}{2}m}\int_0^\pi \{\mu+\sqrt{\mu^2-1}\cos\phi\}^{n-m}\sin^{2m}\phi\, d\phi,$$

or
$$M(1-\mu^2)^{\frac{1}{2}m}\int_0^\pi \frac{\sin^{2m}\phi\, d\phi}{\{\mu+\sqrt{\mu^2-1}\cos\phi\}^{n+m+1}},$$

where
$$M=\frac{(n+m)!}{(n-m)!\,1.3\ldots(2m-1)\,\pi}.$$

Therefore

$$\frac{P'^m_n(\mu')}{r'^{n+1}}=M\varpi^m\int_0^\pi\frac{\sin^{2m}\phi\, d\phi}{(z'+\iota\varpi\cos\phi)^{n+m+1}}$$

$$=M\varpi^m\int_0^\pi\frac{\sin^{2m}\phi\, d\phi}{[c+r\{\mu+\sqrt{\mu^2-1}\cos\phi\}]^{n+m+1}};$$

whence, if $\lambda=\mu+\sqrt{\mu^2-1}\cos\phi$, and $r<c$,

$$\frac{P'^m_n(\mu')}{r'^{n+1}}=\frac{M\varpi^m}{c^{n+m+1}}\int_0^\pi\left\{1-(n+m+1)\frac{r\lambda}{c}+\frac{(n+m+1)(n+m+2)}{2!}\left(\frac{r\lambda}{c}\right)^2\ldots\right.$$
$$\left.\ldots+\frac{(-)^s(n+m+1)\ldots(n+m+s)}{s!}\left(\frac{r\lambda}{c}\right)^s+\ldots\right\}\sin^{2m}\phi\, d\phi;$$

whence, by the first form of P_n^m, we obtain

$$\frac{P'^m_n}{r'^{n+1}}=\frac{r^m}{(n-m)!\,c^{n+m+1}}\left[\frac{(n+m)!}{2m!}P_m^m-\frac{(n+m+1)!}{(2m+1)!}\frac{r}{c}P_{m+1}^m+\ldots\right.$$
$$\left.\ldots+\frac{(-)^s(n+m+s)!}{(2m+s)!}\left(\frac{r}{c}\right)^s P_{m+s}^m+\ldots\right].$$

In order to obtain the second equation, let us change θ and θ' into their supplements; then, since

$$P_n^m\{\cos(\pi-\theta)\}=(-)^{n-m}P_n^m(\cos\theta),$$

we obtain

$$\frac{(-)^{n-m}P_n^m(\mu)}{r^{n+1}}=\frac{r'^m}{(n-m)!\,c^{n+m+1}}\left[\frac{(n+m)!}{2m!}P'^m_m+\frac{(n+m+1)!}{(2m+1)!}\frac{r'}{c}P'^m_{m+1}\right.$$
$$\left.\ldots+\frac{(n+m+s)!}{(2m+s)!}\left(\frac{r'}{c}\right)^s P'^m_{m+s}+\ldots\right].$$

The corresponding formulae when $r>c$ or $r'>c$ could be easily obtained, but they are not required for the present investigation.

[1] These formulae will be proved in the second volume. See also Heine, *Kugelfunctionen*, ch. IV.: *Mess. Math.*, vol. XIII., p. 147.

228. Let ϕ_1 be the velocity potential of the liquid when A is moving with velocity v_1, whilst B is kept at rest, and let ϕ_2 be the velocity potential when B is in motion and A is fixed. Then if ϕ be the velocity potential of the whole motion,

$$\phi = \phi_1 + \phi_2 \qquad\qquad (36).$$

The problem is therefore reduced to the determination of ϕ_1, for when this is known, ϕ_2 can be written down by symmetry.

Let χ be the angle which a plane through AB and any point P makes with the plane through AB which contains the directions of motion of A and B; also let Q_n, Q_n' be written for P_n^1 and P'^1_n. Then, in the neighbourhood of A, ϕ_1 must be expressible in the form of the series

$$\phi_1 = \left\{-\frac{v_1 a^3 Q_1}{2r^2} + A_1\left(r + \frac{a^3}{2r^2}\right)Q_1 + A_2\left(r^2 + \frac{2a^5}{3r^3}\right) + \dots\right\}\cos\chi \quad (37),$$

for this value of ϕ_1 satisfies the surface condition

$$\left(\frac{d\phi_1}{dr}\right)_a = v_1 \sin\theta\cos\chi.$$

In the neighbourhood of B, ϕ_1 must be expressible in the form

$$\phi_1 = \left\{B_1\left(r' + \frac{b^3}{2r'^2}\right)Q_1' + B_2\left(r'^2 + \frac{2b^5}{3r'^3}\right)Q_2' + \dots\right\}\cos\chi \qquad (38),$$

for the value of ϕ_1 satisfies the surface condition

$$\left(\frac{d\phi_1}{dr'}\right)_b = 0.$$

The series consisting of powers of r^{-1} and r'^{-1} are convergent at all points outside the two spheres, but the series consisting of powers of r and r' will be divergent if r and r' be sufficiently great; but we shall only require these latter series in the neighbourhood of the two spheres where they are convergent.

The kinetic energy consists of a series of terms of the form

$$\begin{aligned}\int_0^{2\pi} d\chi \int_0^{\pi} Q_n v_1 a^2 \cos^2\chi \sin^2\theta \, d\theta &= \pi a^2 v_1 \int_{-1}^{1} (1-\mu^2)\frac{dP_n}{d\mu}\,d\mu \\ &= 2\pi a^2 v_1 \int_{-1}^{1} \mu P_n \, d\mu \\ &= \frac{4\pi a^2 v_1}{3}\,(n=1) \qquad (39), \\ &= 0 \quad (n \text{ any other value}).\end{aligned}$$

Hence the terms involving Q_2, Q_3, &c. contribute nothing to the energy, and we may therefore, in writing down the final value of ϕ_1, reject all terms except those involving Q_1 or Q_1'.

229. Dropping the factor $\cos\chi$ for the present, we should have, if B were absent,

$$\phi_1 = -\frac{v_1 a^3 Q_1}{2r^2}.$$

Putting $m=1$, $n=1$ in (35), the value of this near B is

$$\phi_1 = -\frac{v_1 a^3}{2c^3}\left(r' Q_1' + \frac{r'^2 Q_2'}{c} + \frac{r'^3 Q_3'}{c^3} + \frac{r'^4 Q_4'}{c^4} + \dots\right).$$

From (38) it follows that, in order to make the velocity at B vanish, we must add the series

$$-\frac{v_1 a^3}{2c^3}\left(\frac{b^3 Q_1'}{2r'^2} + \frac{2b^5 Q_2'}{3cr'^3} + \frac{3b^7 Q_3'}{4c^2 r'^4} + \frac{4b^9 Q_4'}{5c^3 r'^5} + \dots\right).$$

Transforming this latter series back again to A by (34), and retaining the important terms only, the value of ϕ_1 near A becomes

$$\phi_1 = -\frac{v_1 a^3 Q_1}{2r^2} - \frac{v_1 a^3 b^3}{c^6}\left(\frac{1}{4} + \frac{b^2}{c^2} + \frac{9b^4}{4c^4} + \frac{4b^6}{c^6}\right) Qr + \frac{v_1 a^3 b^3}{4c^7}\, Q_2 r^2.$$

In order to satisfy the surface condition at A, add the terms

$$-\frac{v_1 a^3 b^3}{c^6}\left(\frac{1}{4} + \frac{b^2}{c^2} + \frac{9b^4}{4c^4} + \frac{4b^6}{c^6}\right)\frac{Q_1 a^3}{2r^2} + \frac{v_1 a^8 b^3 Q_2}{6c^7 r^3}.$$

Neglecting powers of c^{-1} higher than the twelfth, the value of these added terms near B is

$$-\frac{v_1 a^6 b^3}{2c^9}\left(\frac{1}{4} + \frac{b^2}{c^2}\right) Q_1' r' - \frac{v_1 a^8 b^3}{2c^{11}}\, Q_1' r'.$$

Adding the terms

$$-\frac{v_1 a^6 b^3}{4c^9}\left(\frac{1}{4} + \frac{b^2}{c^2} + \frac{a^2}{c^2}\right)\frac{Q_1' b^3}{r'^2} \quad \dots\dots\dots\dots\dots (40),$$

omitting Q_2', &c., and restoring $\cos\chi$, the value of the velocity potential near B becomes

$$\phi_1 = -\frac{v_1 a^3}{2c^3}\left\{1 + \frac{a^3 b^3}{4c^6} + \frac{a^3 b^3 (a^2 + b^2)}{c^8}\right\}\left(r' + \frac{b^3}{2r'^2}\right) Q_1' \cos\chi \dots (41).$$

The first term of (40) on transformation becomes

$$-v_1 a^6 b^6 Q_1 r / 16c^{12},$$

whence the value of ϕ_1 near A is

$$\phi_1 = -\frac{v_1 a^3 Q_1}{2r^2}\cos\chi - \frac{v_1 a^3 b^3}{c^6}\left\{\frac{1}{4} + \frac{b^2}{c^2} + \frac{9b^4}{4c^4} + \frac{(a^3 + 64b^3)\, b^3}{16c^6}\right\}$$
$$\times\left(r + \frac{a^3}{2r^2}\right) Q_1 \cos\chi \ldots\ldots\ldots\ldots\ldots\ldots (42).$$

The values of ϕ_2 at A and B can be written down by symmetry; whence, if T be the kinetic energy of the system

$$2T = A'v_1^2 + 2B'v_1v_2 + C'v_2^2,$$

where

$$\left.\begin{aligned} A' &= m_1 - \rho\iint \phi_1 \frac{d\phi_1}{dn}\, dS_1 \\ &= m_1 + \tfrac{1}{2}M_1\left[1 + \frac{3a^3b^3}{c^6}\left\{\frac{1}{4} + \frac{b^2}{c^2} + \frac{9b^4}{4c^4} + \frac{b^3(a^3 + 64b^3)}{16c^6}\right\}\right] \\ C' &= m_2 + \tfrac{1}{2}M_2\left[1 + \frac{3a^3b^3}{c^6}\left\{\frac{1}{4} + \frac{a^2}{c^2} + \frac{9a^4}{4c^4} + \frac{a^3(b^3 + 64a^3)}{16c^6}\right\}\right] \\ B' &= -\rho\iint \phi_1 \frac{d\phi_2}{dn}\, dS_2 \\ &= \frac{\pi\rho a^3b^3}{c^3}\left\{1 + \frac{a^3b^3}{4c^6} + \frac{a^3b^3(a^2 + b^2)}{c^8}\right\} \end{aligned}\right\} \ldots (43),$$

where m_1, m_2 are the masses of the spheres A and B; M_1, M_2 those of the liquid displaced by them, and ρ is the density of the liquid.

The values of A', B', C' have been calculated by Mr Herman as far as c^{-15}.

230. We shall now apply the preceding results to obtain the solution of some problems.

If a sphere is projected in a liquid which is bounded by a fixed plane, we must put $a = b$, $u_1 = -u_2 = u$, $v_1 = v_2 = v$; then

$$2T = (A + B)\, u^2 + (A' + B')\, v^2,$$

and, if higher powers than c^{-6} be neglected, we obtain from (24) and (43)

$$\left.\begin{aligned} A + B &= m + \tfrac{1}{2}M\left(1 + \frac{3a^3}{c^3} + \frac{3a^6}{c^6}\right) \\ A' + B' &= m + \tfrac{1}{2}M\left(1 + \frac{3a^3}{2c^3} + \frac{3a^6}{4c^6}\right) \end{aligned}\right\} \ldots\ldots\ldots (44),$$

where $\frac{1}{2}c$ is the distance of the sphere from the plane. Lagrange's equation

$$\frac{d}{dt}\frac{dT}{du} - 2\frac{dT}{dc} = 0$$

gives
$$(A+B)\,\dot{u} = v^2\frac{d}{dc}(A'+B') - u^2\frac{d}{dc}(A+B).$$

Also, since the momentum parallel to the plane is constant

$$(A'+B')\,v = \text{const.} = G.$$

Let V be the resultant velocity of the sphere, θ the angle which its direction makes with the normal to the plane, then

$$(A+B)\,\dot{u} = V^2\left\{\sin^2\theta\,\frac{d}{dc}(A'+B') - \cos^2\theta\,\frac{d}{dc}(A+B)\right\}$$
$$= \frac{9MV^2a^3}{2c^4}\left\{\left(1+\frac{2a^3}{c^3}\right)\cos^2\theta - \tfrac{1}{2}\left(1+\frac{a^3}{c^3}\right)\sin^2\theta\right\}.$$

If, therefore,

$$\tan\alpha = \sqrt{\frac{2\,(c^3+2a^3)}{c^3+a^3}};$$

it follows that, whenever the direction of motion makes with the normal to the plane an angle which is $< \alpha$ or $> \pi - \alpha$, the sphere will be repelled from the plane; but, whenever this angle lies between α and $\pi - \alpha$, the sphere will be attracted. Also, since $A' + B'$ increases as c diminishes, the velocity parallel to the plane will be accelerated when the direction of motion lies between α and $\pi - \alpha$; and retarded when this direction makes with the normal an angle $< \alpha$ or $> \pi - \alpha$. If, therefore, the sphere be projected parallel to the plane, it will ultimately strike it.

We have shown in § 208 that in the case of a cylinder $\alpha = \frac{1}{4}\pi$, hence in the case of a sphere $\alpha > \frac{1}{4}\pi$. The discussion of the subsequent motion of a sphere projected in any given direction in a liquid bounded by a fixed plane, can be carried on in the same manner as in the corresponding case of a cylinder, but it must be recollected that the preceding values of the coefficients may not give correct results if the sphere gets too close to the plane.

231. Let X, Y be the forces upon the sphere, arising from the pressure of the liquid, then

$$X = m\dot{u} = m\left\{v^2\frac{d}{dc}(A'+B') - u^2\frac{d}{dc}(A+B)\right\}.\,/(A+B),$$
$$Y = m\dot{v} = -2muv\,\frac{d}{dc}(A'+B').\,/(A'+B').$$

From (44) we obtain

$$\frac{d}{dc}(A+B) = -\frac{9Ma^3}{2c^4}\left(1+\frac{2a^3}{c^3}\right),$$

$$\frac{d}{dc}(A'+B') = -\frac{9Ma^3}{4c^4}\left(1+\frac{a^3}{c^3}\right),$$

whence neglecting higher powers than c^{-7} we obtain

$$X = \frac{9Mma^3}{(2m+M)c^4}\left\{u^2 - \tfrac{1}{2}v^2 + \frac{a^3\left[u^2(4m-M)+v^2(M-m)\right]}{(2m+M)c^3}\right\},$$

$$Y = \frac{9Mmuva^3}{(2m+M)c^4}\left\{1+\frac{(4m-M)a^3}{2(2m+M)c^3}\right\}.$$

232. Let us now suppose that the sphere A is a pendulum performing small oscillations along AB about its mean position, whilst the sphere B is free to move.

Let A be the mean position of A, B the initial position of B: A', B' their displaced positions, and let $AA' = x$, $BB' = y$, $AB = c$; $A'B' = p$. Then $p = c + x - y$ and if $-\mu x$ is the force required to maintain the oscillation of A, the equations of motion are

$$A'\ddot{x} - B'\ddot{y} - (\dot{x}\dot{y} - \tfrac{1}{2}\dot{x}^2)\frac{dA'}{dp} + \dot{y}^2\frac{dB'}{dp} - \tfrac{1}{2}\dot{y}^2\frac{dC'}{dp} + \mu x = 0,$$

$$C'\ddot{y} - B'\ddot{x} + \tfrac{1}{2}\dot{x}^2\frac{dA'}{dp} - \dot{x}^2\frac{dB'}{dp} - (\tfrac{1}{2}\dot{y}^2 - \dot{x}\dot{y})\frac{dC'}{dp} = 0,$$

where the accents denote the values of A, B, C at time t.

To obtain a first approximation, neglect squares and products of small quantities, and we find

$$(AC - B^2)\ddot{x} + \mu C x = 0,$$

$$C\ddot{y} - B\ddot{x} = 0.$$

If therefore the sphere A is initially displaced to a distance x_0 and then let go, the integrals are

$$x = x_0 \cos kt,$$

$$y = \frac{Bx_0}{C}(\cos kt - 1),$$

where $k^2 = \mu C/(AC - B^2)$.

Since y is negative and increases numerically so long as x lies between x_0 and $-x_0$, it follows that to a first approximation B is repelled from A so long as A is moving away from its initial position A', and attracted when A is returning to A'.

233. In order to obtain a second approximation, we must take into account the squares of small quantities. Let

$$y = \frac{Bx_0}{C}(\cos kt - 1) + z,$$

where z is at least of the order x_0^2. Then

$$p = c - \left(\frac{B}{C} - 1\right) x_0 \cos kt + \frac{Bx_0}{C} + z,$$

$$A' = A + (x - y)\frac{dA}{dc} \text{ \&c.}$$

Therefore B's equation of motion becomes

$$\left\{C + (x - y)\frac{dC}{dc}\right\}\left\{\ddot{z} - \frac{Bx_0k^2}{C}\cos kt\right\}$$

$$+ \left\{B + (x - y)\frac{dB}{dc}\right\} x_0 k^2 \cos kt - x_0^2 k^2 \left(\frac{dB}{dc} - \tfrac{1}{2}\frac{dA}{dc}\right)\sin^2 kt$$

$$- x_0^2 k^2 \left(\frac{B^2}{2C^2} - \frac{B}{C}\right)\frac{dC}{dc}\sin^2 kt = 0.$$

Neglecting cubes of small quantities, this equation may be written

$$C\ddot{z} = f + L\cos kt + M\cos 2kt,$$

where
$$f = -\frac{k^2 x_0^2}{4}\left(\frac{dA}{dc} - \frac{2B}{C}\frac{dB}{dc} + \frac{B^2}{C^2}\frac{dC}{dc}\right)$$

$$= -\frac{k^2 x_0^2}{4}\frac{d}{dc}\left(A - \frac{B^2}{C}\right).$$

If we only take into account the first terms in A and B, which is equivalent to neglecting the twelfth and higher powers of c^{-1}, we obtain from (4) and (21)

$$A = m_1 + \tfrac{1}{2}M_1 + \frac{3M_1\mu_1}{2\mu_0}$$

$$= m_1 + \tfrac{1}{2}M_1 + \frac{2\pi\rho a^6 b^3}{(c^2 - b^2)^3},$$

therefore
$$\frac{dA}{dc} = -\frac{12\pi\rho a^6 b^3 c}{(c^2 - b^2)^4},$$

$$\frac{B^2}{C} = \frac{6\pi\rho^2 a^6 b^3}{(2\sigma + \rho)c^6},$$

therefore
$$\frac{d}{dc}\frac{B^2}{C} = -\frac{36\pi\rho^2 a^6 b^3}{(2\sigma + \rho)c^7},$$

where σ is the density of the sphere B; whence

$$f = 3\pi\rho a^6 b^3 k^2 x_0^2 \left\{\frac{c}{(c^2-b^2)^4} - \frac{3\rho}{(2\sigma+\rho)c^7}\right\}.$$

The term f indicates that the sphere B, in addition to its vibratory motion, will be attracted towards or repelled from the sphere A, according as f is positive or negative. Hence there will be repulsion when

$$\frac{3\rho}{2\sigma+\rho} > \frac{c^8}{(c^2-b^2)^4},$$

i.e. when
$$c > \frac{b}{\{1-\sqrt[4]{\frac{1}{3}(2\sigma/\rho+1)}\}^{\frac{1}{2}}},$$

which can only happen when $\sigma < \rho$ or the density of the sphere B is less than that of the liquid.

If therefore the sphere B is denser than the liquid it will in general be attracted, but if the density of the sphere is less than that of the liquid there will be a critical point, beyond which there will be repulsion, and within which there will be attraction, this critical distance is given by

$$c = \frac{b}{\{1-\sqrt[4]{\frac{1}{3}(2\sigma/\rho+1)}\}^{\frac{1}{2}}}.$$

Since this result has been obtained on this assumption that c is so large compared with a and b, that powers of c^{-1} above the twelfth may be neglected, it fails to give a correct result if with a given density, c comes out nearly equal to b. If $\sigma/\rho = \cdot 9$ then $c = 7\cdot 648b$.

This theorem is due to Sir W. Thomson; the preceding demonstration is due to Mr Hicks.

On the Pulsations of Two Spheres.

234. The term pulsation is applied to denote a periodic change of volume; and the problem which we shall now investigate is the following:—Let there be two spheres in a liquid, whose centres are fixed, and which are composed of some elastic material such as india rubber; let each sphere be compressed or expanded into a concentric sphere and then let go; it is required to determine the motion.

If the spheres were composed of some highly elastic material, the inequality of the pressure of the liquid upon their surfaces would produce a deformation which would cause their surfaces to cease to be spherical; we shall therefore suppose the rigidity of the spheres to be sufficiently great to render such deformations inappreciable.

235. If ϕ_1 be the velocity potential of the liquid when the sphere A pulsates, and B does not; and ϕ_2 be the similar quantity when A and B are interchanged,

$$\phi = \phi_1 + \phi_2.$$

Let a and b be the radii of the spheres A and B, c the distance between their centres. If B were absent the value of ϕ_1 would be $-a^2\dot{a}/r$, for this value of ϕ_1 satisfies the boundary condition $d\phi_1/dr = \dot{a}$. This is the velocity potential due to a source of strength $a^2\dot{a}$ situated at the centre of A, and by § 52 the image of this in B will be a source of strength $a^2b\dot{a}/c$ at the inverse point P, together with a line sink extending from the inverse point to the centre of B, of strength $a^2\dot{a}/b$ per unit of length. Putting $m = a^2b\dot{a}/c$, $f = b^2/c$, the strength of the source at P is m, and that of the line sink from B to P is $-m/f$ per unit of length: and by § 55 the image of these in A is an arrangement of the same kind. Hence ϕ_1 and ϕ_2 will be the velocity potentials of two infinite systems of sources and line sinks, which respectively lie within each of the spheres.

236. Taking the density of the liquid as unity, let F_2 be the resultant of the pressure of the liquid on B towards A, then

$$\begin{aligned} F_2 &= - \iint p \cos\theta dS \\ &= \pi b^2 \int_0^\pi (\dot{\phi} + \tfrac{1}{2} V^2) \sin 2\theta d\theta, \end{aligned}$$

where V is the velocity of the liquid at the surface of B; let

$$P = \int_0^\pi \phi \sin 2\theta d\theta,$$

$$Q = \int_0^\pi \phi^2 \sin 2\theta d\theta.$$

Then $$\int_0^\pi \dot{\phi} \sin 2\theta = \dot{P}.$$

In order to find the portion of F_2 which depends upon V^2, let $v = b^2\dot{b}$, then $V^2 = v^2/b^4 + (d\phi/bd\theta)^2$; and since v is constant over the surface of B, the portion of F_2 depending upon it is zero, whence, denoting the portion of the pressure depending upon V^2 by I, we have

$$I = \tfrac{1}{2}\pi b^2 \int_0^\pi V^2 \sin 2\theta d\theta = \tfrac{1}{2}\pi \int_0^\pi \left(\frac{d\phi}{d\theta}\right)^2 \sin 2\theta d\theta$$

$$= -\tfrac{1}{2}\pi \int_0^\pi \left(\phi \frac{d^2\phi}{d\theta^2} \sin 2\theta + \frac{d\phi^2}{d\theta} \cos 2\theta\right) d\theta$$

$$= -\tfrac{1}{2}\pi \int_0^\pi \phi \left(\frac{d^2\phi}{d\theta^2} + 2\phi\right) \sin 2\theta\, d\theta - \tfrac{1}{2}\pi \left[\phi^2\right]_0^\pi.$$

By Laplace's equation

$$-\frac{d^2\phi}{d\theta^2} = 2b\frac{d\phi}{dr} + b^2 \frac{d^2\phi}{dr^2} + \cot\theta \frac{d\phi}{d\theta}$$

in which r is to be put equal to b after the differentiations have been performed. Hence $d\phi/dr = v/b^2$, so that

$$I = \tfrac{1}{2}\pi \int_0^\pi \phi \left\{\left(\frac{2v}{b} + b^2 \frac{d^2\phi}{dr^2} - 2\phi\right) \sin 2\theta + 2\cos^2\theta \frac{d\phi}{d\theta}\right\} d\theta - \tfrac{1}{2}\pi \left[\phi^2\right]_0^\pi.$$

But

$$2\int_0^\pi \cos^2\theta\, \phi \frac{d\phi}{d\theta} d\theta = \int_0^\pi \phi^2 \sin 2\theta d\theta + \left[\phi^2\right]_0^\pi = Q + \left[\phi^2\right]_0^\pi,$$

and

$$\int_0^\pi \phi \frac{d^2\phi}{dr^2} \sin 2\theta d\theta = \int_0^\pi \left(\tfrac{1}{2} \frac{d^2\phi^2}{dr^2} - \frac{v^2}{b^4}\right) \sin 2\theta d\theta = \tfrac{1}{2} \frac{d^2 Q}{dr^2} \quad \ldots \quad (45)$$

whence $$I = \tfrac{1}{2}\pi \left(\frac{2v}{b} P - Q + \tfrac{1}{2} b^2 \frac{d^2 Q}{dr^2}\right),$$

and $$F_2 = \pi \left(b^2 \dot{P} + \frac{v}{b} P - \tfrac{1}{2} Q + \tfrac{1}{4} b^2 \frac{d^2 Q}{dr^2}\right) \quad \ldots\ldots\ldots\ldots (46)$$

when $r = b$.

237. Let P_1 be the part of P due to ϕ_1, then if μ_n be the strength of any image whose distance from B is r, the portion of P_1 due to this is

$$-2\int_0^\pi \frac{\mu_n \sin\theta \cos\theta d\theta}{(b^2 + r^2 - 2br\cos\theta)^{\frac{1}{2}}}$$

which is equal to

$$-\frac{4\mu_n b}{3r^2},\ r > b;\ \text{and} -\frac{4\mu_n r}{3b^2},\ r < b.$$

Hence if μ_n be the strength of the nth source image in A from A, and ρ'_n that of the other extremity of the line sink image; the part of P_1 due to μ_n is

$$X_n = -\frac{4\mu_n b}{3(c-\rho_n)^2} + \tfrac{4}{3}\int_{\rho'_n}^{\rho_n} \frac{\mu_n b\,dx}{(\rho_n - \rho'_n)(c-x)^2}$$

$$= -\frac{4\mu_n b(\rho_n - \rho'_n)}{3(c-\rho_n)^2(c-\rho'_n)} \quad\text{.........................(47).}$$

Let ν_n denote the strength of the nth image in B, σ_n, σ'_n the distances of its extremities from B; then the part of P_1 due to ν_n is

$$X_n' = -\frac{4\nu_n\sigma_n}{3b^2} + \tfrac{4}{3}\int_{\sigma'_n}^{\sigma_n} \frac{\nu_n x\,dx}{(\sigma_n - \sigma'_n)b^2}$$

$$= -\frac{2\nu_n(\sigma_n - \sigma'_n)}{3b^2} \quad\text{.............................(48).}$$

Now

$$\nu_n = \frac{b\mu_n}{c-\rho_n},\quad \sigma_n = \frac{b^2}{c-\rho_n},\quad \sigma'_n = \frac{b^2}{c-\rho'_n},$$

therefore
$$X_n' = -\frac{2\mu_n b(\rho_n - \rho'_n)}{3(c-\rho_n)^2(c-\rho'_n)} \quad\text{..............(49).}$$

Adding (47) and (49) and summing for all integral values of n from ∞ to 0, we obtain

$$P_1 = -2\Sigma_0^\infty \frac{\mu_n b(\rho_n - \rho'_n)}{(c-\rho_n)^2(c-\rho'_n)} \quad\text{..............(50).}$$

238. In order to find the portion P_2 of P due to ϕ_2, we must remember that the original source is now in B. Let σ_n, σ'_n denote the distances of the extremities of the nth image in B from B, due to ϕ_2, then expressing μ_n, ρ_n, ρ'_n in terms of ν_n, σ_n, σ'_n we shall obtain

$$P_2 = -2\Sigma_0^\infty \frac{\nu_n(\sigma_n - \sigma_n')}{b^2}.$$

Hence $$P = -2\Sigma_0^\infty \frac{\mu_n b(\rho_n - \rho'_n)}{(c-\rho_n)^2(c-\rho'_n)} - 2\Sigma_0^\infty \frac{\nu_n(\sigma_n - \sigma'_n)}{b^2} \quad\text{...(51),}$$

where μ_n, ρ_n, ρ'_n refer to the images of A's motion, and ν_n, σ_n, σ'_n to those of B's motion.

By direct calculation we easily find

$$\left.\begin{aligned}
\rho_0 &= 0, & \rho'_0 &= \infty \\
\rho_1 &= \frac{a^2c}{c^2 - b^2}, & \rho'_1 &= \frac{a^2}{c} \\
\rho_2 &= \frac{a^2c\,(c^2 - a^2 - b^2)}{(c^2 - b^2)^2 - a^2c^2}, & \rho'_2 &= \frac{a^2\,(c^2 - a^2)}{c\,(c^2 - a^2 - b^2)}
\end{aligned}\right\}\ldots(52),$$

also if m_1 be the mass of the liquid displaced by A,

$$\mu_0 = a^2\dot{a} = \frac{\dot{m}_1}{4\pi}, \quad \mu_1 = \frac{ab\dot{m}_1}{4\pi\,(c^2 - b^2)}, \quad \mu_2 = \frac{a^2b^2\dot{m}_1}{4\pi\,\{(c^2 - b^2)^2 - a^2c^2\}}\ \ldots(53).$$

The ν's and σ's can be obtained by symmetrically interchanging a and b and putting m_2 for m_1. If we write M_2, N_2 for the two series in the right-hand side of (51), we shall find that

$$M_2 = \frac{\dot{m}_1 b}{4\pi c^2}\left[1 + \frac{a^3b^3}{(c^2 - a^2)(c^2 - a^2 - b^2)^2}\right.$$
$$\left. + \frac{a^6b^6}{\{(c^2 - a^2)^2 - b^2c^2\}\{(c^2 - b^2)^2 - 2a^2c^2 + a^2(a^2 + b^2)\}^2} + \ldots\right]\ldots\ldots(54),$$

$$N_2 = \frac{\dot{m}_2 a^3 b}{4\pi c}\left[\frac{1}{(c^2 - a^2)^2}\right.$$
$$\left. + \frac{a^3b^3}{(c^2 - a^2 - b^2)\{(c^2 - a^2)^2 - b^2c^2\}^2} + \ldots\right]\ldots(55),$$

and
$$P = -2\,(M_2 + N_2)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(56).$$

From the above formulae it appears that M_2 is of the order c^{-2}, and N_2 of the order c^{-5}.

239. The value of the portion of F_2 which depends on the square of the velocity is more difficult to obtain, and we shall content ourselves with obtaining an approximate value as far as the term c^{-5}.

Let us now put $u = a^2\dot{a}$, $v = b^2\dot{b}$, and let P_n denote zonal harmonics when the origin is at A and axis BA, and P'_n similar quantities when the origin is at B.

Near the surface of B

$$\phi_1 = \Sigma_1^\infty A_n\left\{R^n + \frac{b^{2n+1}}{(n+1)\,R^{n+1}}\right\}P'_n + \text{const.},$$

and
$$\phi_2 = -\frac{v}{R} + \Sigma_1^\infty B_n\left\{R^n + \frac{b^{2n+1}}{(n+1)\,R^{n+1}}\right\}P'_n + \text{const.}$$

Dropping the accents for the present and writing C_n for the coefficient of P_n^2 in the value of ϕ, we obtain

$$Q = 2\int_{-1}^{1} (\Sigma C_n P_n)^2 \mu d\mu.$$

Since P_n^2 is unchanged when $-\mu$ is written for μ,

$$\int_{-1}^{1} P^2{}_n \mu d\mu = 0.$$

Hence
$$Q = 4\Sigma C_m C_n \int_{-1}^{1} P_m P_n \mu d\mu,$$

the summation extending to all positive integral values of m, n except $m = n$. Let

$$\Phi = \int_{\mu}^{1} P_m P_n d\mu.$$

Then
$$\int_{-1}^{1} P_m P_n \mu d\mu = \int_{-1}^{1} P_m P_n d\mu + \int_{-1}^{1} \Phi d\mu$$
$$= \int_{-1}^{1} \Phi d\mu.$$

Now (Ferrers' *Spherical Harmonics*, § 24),

$$\Phi = \frac{1}{(m-n)(m+n+1)} \left\{ \frac{n(n+1)}{2n+1} P_m (P_{n+1} - P_{n-1}) - \frac{m(m+1)}{2m+1} P_n (P_{m+1} - P_{m-1}) \right\}.$$

Hence $\int_{-1}^{1} \Phi d\mu$ vanishes unless $m = n+1$ or $n-1$ and its values in the two cases are

$$\frac{2(m+1)}{(2m+1)(2m+3)} \text{ and } \frac{2m}{(2m-1)(2m+1)}.$$

240. Putting $m = 0$, $r' = R$ in (34) and (35) we obtain

$$\left. \begin{aligned} \frac{P'_n}{R^{n+1}} &= \frac{P_0}{c^{n+1}} - \frac{(n+1)P_1 r}{c^{n+2}} + \frac{(n+1)(n+2)P_2 r^2}{2!\, c^{n+3}} - \dots \\ \frac{(-)^n P_n}{r^{n+1}} &= \frac{P'_0}{c^{n+1}} + \frac{(n+1)P'_1 R}{c^{n+2}} + \frac{(n+1)(n+2)P_2 R^2}{2!\, c^{n+3}} + \dots \end{aligned} \right\} \dots (57).$$

Now if B were absent, the value of ϕ_1 would be

$$\phi_1 = -\frac{u}{r}.$$

The value of this near B is

$$\phi_1 = -\frac{u}{c}\left(P_0' + \frac{P_1'R}{c} + \frac{P_2'R^2}{c^2} + \dots\right).$$

In order to make the velocity at the surface of B vanish, we must add the series

$$-\frac{ub^3}{c^2}\left(\frac{P_1'}{2R^2} + \frac{2P_2'b^2}{3cR^3} + \frac{3P_3'b^4}{4c^2R^4} + \dots\right).$$

Transforming each term of the last series by means of (57), the value of ϕ_1 near A becomes

$$\phi_1 = -\frac{u}{r} - \frac{ub^3}{2c^4}\left\{P_0 - \frac{P_1 r}{c} + \dots\right\}.$$

Adding the proper series, the value of ϕ_1 near A becomes

$$\phi_1 = -\frac{u}{r} - \frac{ub^3}{2c^4} + \frac{ub^3}{2c^5}\left(r + \frac{a^3}{2r^2}\right)P_1 + \quad \dots\dots\dots\dots(58).$$

The added term produces at B a constant term of the order c^{-7}, which contributes nothing to the pressure, hence the value of ϕ_1 near B is

$$\phi_1 = -\frac{u}{c} - \frac{u}{c^2}\left(R + \frac{b^3}{2R^2}\right)P_1' - \frac{u}{c^3}\left(R^2 + \frac{2b^5}{3R^3}\right)P_2' \dots\dots(59).$$

Changing P_1 into $-P_1'$, it follows from (58) that the value of ϕ_2 near B is

$$\phi_2 = -\frac{v}{R} - \frac{va^3}{2c^4} - \frac{va^3}{2c^5}\left(R + \frac{b^3}{2R^2}\right)P_1' \dots\dots\dots\dots(60),$$

whence the value of ϕ near B is

$$\phi = -\frac{v}{R} - \frac{u}{c} - \frac{va^3}{2c^4} - \left(\frac{u}{c^2} + \frac{va^3}{2c^5}\right)\left(R + \frac{b^3}{2R^2}\right)P_1' - \frac{u}{c^3}\left(R^2 + \frac{2b^5}{3R^3}\right)P_2 \\ - \&c. \dots$$

Putting in this $R = b$, we obtain

$$\phi_b = -\frac{v}{b} - \frac{u}{c} - \frac{va^3}{2c^4} - \frac{3b}{2}\left(\frac{u}{c^2} + \frac{va^3}{2c^5}\right)P_1' - \frac{5ub^2}{3c^3}P_2' - .$$

also $$\left(\frac{d^2\phi}{dR^2}\right)_b = -\frac{2v}{b^3} - \frac{3}{b}\left(\frac{u}{c^2} + \frac{va^3}{2c^5}\right)P_1' - \frac{10u}{c^3}P_2' - .$$

Therefore $$Q = 2\int_{-1}^{1}\phi^2\mu d\mu$$

$$= 4\left(\frac{uv}{c^2} + \frac{u^2b}{c^3} + \frac{v^2a^3}{2c^5}\right) + \frac{8u^2b^2}{3c^5}.$$

Also by (45)

$$\frac{d^2Q}{dR^2} = 4\int_{-1}^{1} \phi \frac{d^2\phi}{dR^2} \mu d\mu$$

$$= 8\left\{\frac{2uv}{b^2c^2} + \frac{u^2}{bc^3} + \frac{v^2a^3}{b^2c^5} + \frac{8u^2b}{3c^5}\right\}.$$

Restoring the values of u and v, we obtain

$$\frac{b^2}{4}\frac{d^2Q}{dR^2} - \tfrac{1}{2}Q = a^2b^2\left(\frac{2\dot{a}\dot{b}}{c^2} + \frac{ab^2\dot{b}^2}{c^5}\right).$$

By (54), (55) and (56)

$$P = -\frac{2a^2b\ddot{a}}{c^2} - \frac{2a^3b^3\ddot{b}}{c^5} - \text{higher powers of } c^{-1}.$$

Therefore by (46) the force depending on the square of the velocity

$$= -\frac{\pi a^3b^4\dot{b}^2}{c^5} \text{ ....(61)},$$

which varies as c^{-5}.

Hence $$F_2 = -2\pi b^2\frac{d}{dt}(M_2 + N_2) - \frac{a^3b^4\dot{b}^2}{c^5}.$$

The value of F_1 the force on A *towards* B, can be obtained by symmetrically interchanging a and b.

241. If we neglect all powers of c^{-1} above the second

$$F_2 = -\frac{2\pi b^2}{c^2}\frac{d}{dt}(a^2b\dot{a}).$$

Let $$a = \bar{a} + \alpha \sin\frac{2\pi t}{T},$$

$$b = \bar{b} + \beta \sin\frac{2\pi}{T}(t - \epsilon),$$

so that $\bar{a}$, $\bar{b}$ denote the mean values of the radii. The mean value of F_2 will be

$$\bar{F}_2 = -\frac{2\pi}{Tc^2}\int_0^T b^2\frac{d}{dt}(a^2\dot{a}b)\,dt$$

$$= \frac{4\pi}{Tc^2}\int_0^T a^2b^2\dot{a}\dot{b}\,dt$$

$$= \frac{16\pi^3}{T^3c^2}(\bar{a}\bar{b})^2\alpha\beta\int_0^T \cos\frac{2\pi t}{T}\cos\frac{2\pi}{T}(t-\epsilon)\,dt$$

$$= \frac{8\pi^3\bar{a}^2\bar{b}^2\alpha\beta}{T^2c^2}\cos\frac{2\pi\epsilon}{T} = \bar{F}_1.$$

Hence if the spheres are pulsating in the same periodic time *they will attract one another when their phases differ by less than a quarter of a period; but if the phases differ by more than a quarter and less than three quarters of a period, they will repel one another.*

EXAMPLES.

1. An infinite liquid contains a fixed sphere of radius b, and a sphere of radius a and mass M fastened to a spiral spring performing small oscillations in the line joining the spring to the centre of the sphere. Prove that if a and b are so small (or c so large) that we may neglect powers of a/c and b/c above the sixth, the time of oscillation is

$$T\left\{1+\tfrac{3}{2}\frac{M_1}{2M+M_1}\left(\frac{ab}{c^2}\right)^3\right\}^{\frac{1}{2}},$$

where M_1 is the mass of the liquid displaced by the moving sphere, T the time of oscillation if the fixed sphere were removed from the liquid, and c the mean distance between the centres of the spheres.

2. An infinite mass of liquid is divided into two parts by an infinite rigid plane, and a sphere is moving in the liquid in a line perpendicular to the plane. Explain by general reasoning what will be the effect of making a circular opening in the plane with its centre in the line of motion of the sphere, when the sphere is moving (i) towards the plane, (ii) from the plane.

3. Two equal small spheres of mass m and radius a, which attract each other with a force equal to the product of their masses divided by the square of the distance between them, move in a straight line towards each other in an infinite liquid. If λ is the ratio of the density of the liquid to that of the spheres, and x the distance between their centres; prove that so long as $(a/x)^4$ and higher powers can be neglected, the velocity of either sphere is

$$\frac{x\sqrt{m\left(1+\frac{1}{2}\lambda\right)}}{\left\{x^3+\frac{1}{2}\lambda\left(x^3+3a^3\right)\right\}^{\frac{1}{2}}},$$

the motion beginning when the spheres are at an infinite distance apart.

4. If a spherical vessel of radius a contain a concentric sphere of radius b and density σ, the intermediate space being filled with liquid of density ρ, prove that if the vessel be moved with velocity U, the concentric sphere will move forward with relative velocity

$$\frac{(\rho-\sigma)\,U}{\frac{1}{2}\rho\,(a^3+2b^3)/(a^3-b^3)+\sigma}.$$

5. An impulse I is applied to one of two spheres, perpendicular to the line joining their centres. Prove that with the notation of § 229, both spheres will begin to move parallel to the direction of the impulse and in opposite directions, and that their velocities v_1, v_2 are determined by the equations

$$\frac{v_1}{C'}=-\frac{v_2}{B'}=\frac{I}{A'C'-B'^2}.$$

6. Liquid of unit density fills the space between two concentric spheres. The outer one whose radius is b and the inner one whose radius is a, is suddenly distorted in such a manner that the velocity at any point of its surface is $cF(\theta,\,\phi)$, with the condition that its volume remains unaltered. Find the velocity potential of the liquid, and prove that when $F(\theta,\phi)$ is a zonal harmonic of degree n, the kinetic energy of the liquid is

$$\frac{2a^3\,\{nb^{2n+1}+(n+1)\,a^{2n+1}\}\,\pi c^2}{n\,(n+1)\,(2n+1)\,(b^{2n+1}-a^{2n+1})}.$$

7. Liquid is confined within a sphere of radius b; and a solid sphere of radius a is moving with velocity v along a radius of the fixed sphere. Prove that if the distance x between the centres of the two spheres is small compared with b, the velocity potential is approximately equal to

$$-\tfrac{1}{2}ua^3\left\{\left(\frac{1}{r^2}+\frac{2r}{b^3}\right)\cos\theta+x\left(\frac{1}{r^3}+\frac{3r^2}{2b^5}\right)(3\cos^2\theta-1)\right\},$$

the origin being the centre of the fixed sphere.

8. The space between a spherical envelope and a solid concentric sphere is filled with liquid which is at rest. If the outer surface is moved so that at each point its velocity is a spherical surface harmonic Y_n, prove that the solid sphere will remain at rest, unless $n=1$.

9. Prove that the augmented inertia of a ball pendulum of radius a oscillating in a spherical envelope of radius b is

$$\tfrac{1}{2}M(2a^3+b^3)/(b^3-a^3)$$

where M is the mass of the liquid displaced.

10. A string of length $f-a$ is attached to a sphere of radius a and mass m, by means of some mechanical arrangement which prevents the sphere from rotating. The other end of the string is attached to a fixed point, and the system is surrounded by a liquid of unlimited extent, which is bounded by a fixed plane. Prove that if the string is initially at right angles to the plane, and sphere is projected perpendicularly to the string, with velocity V, the tension of the latter will be equal to

$$\frac{m}{f}\left[1-\frac{3Ma^3}{2(2m+M)c^3}\left\{\frac{3f}{c}(1-\cos\theta)+\left(1-\frac{3f}{c}\cos\theta\right)\sin^2\theta\right\}\right]V^2 - \frac{9MV^2ma^3\cos^2\theta}{2(2m+M)c^4},$$

where $\tfrac{1}{2}c$ is the distance of the fixed point from the plane, θ the angle which the string makes with its initial position, M the mass of the liquid displaced by the sphere, and powers higher than c^{-4} are neglected.

APPENDIX.

I. *To prove the equation*

$$p = k\rho^\gamma.$$

The laws of Boyle and Charles show that the pressure, volume, and temperature of a gas are connected by the relation

$$pv = R\theta \quad \text{.................................(1)},$$

where R is a constant, and θ is the temperature measured from the absolute zero of the air thermometer, i.e. from -270° C.

Let a quantity dH of heat be communicated to the gas; the effect of communicating this amount of heat will be to change the pressure, volume, and temperature of the gas, and since by (1) the volume is a function of the pressure and temperature we may put

$$dH = K_p d\theta + \lambda dp \quad \text{..........................(2)},$$

where K_p is the specific heat at constant pressure. From (1) we have

$$\frac{d\theta}{\theta} = \frac{dp}{p} + \frac{dv}{v} \quad \text{...........................(3)},$$

whence eliminating dp from (2) we obtain

$$dH = K_p d\theta + \lambda p \left(\frac{d\theta}{\theta} - \frac{dv}{v}\right),$$

whence if K_v be the specific heat at constant volume

$$K_v = K_p + \frac{\lambda p}{\theta} \quad \text{..............................(4)}.$$

Let us now suppose that the gas experiences a small change of volume but without loss or gain of heat, then $dH = 0$, and (2) becomes

$$K_p d\theta + \lambda dp = 0.$$

Eliminating θ and λ by means of (3) and (4), and putting $\gamma = K_p/K_v$, we obtain

$$\frac{dp}{p} + \gamma \frac{dv}{v} = 0 \quad \text{...........................(5)}.$$

Now it is an experimental fact that γ is independent of the pressure, temperature or volume, whence integrating (5) we obtain

$$pv^\gamma = \text{const.},$$

or

$$p = k\rho^\gamma,$$

where ρ is the density.

II. *To express the value of R (see page 220) in terms of elliptic functions.*

The value of R is

$$R = \pi a^2 \left\{1 + 2\Sigma_1^\infty \frac{(1-q)^2 q^m}{(1-q^{m+1})^2}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

and we have to express this series in terms of elliptic functions. From § 124 it follows that the value of R or $(P+L)\rho^{-1}$ is

$$4\pi c^2 \Sigma_1^\infty \frac{nq^n(1+q^{2n}) + 2nq^{2n}}{1-q^{2n}} = 4\pi c^2 \Sigma_1^\infty \frac{n(1+q^n)q^n}{1-q^n}$$

$$= 4\pi c^2 \Sigma_1^\infty n\left\{\frac{1+q^n}{1-q^n} - 1 - q^n\right\} \ldots (2).$$

Now $\quad \dfrac{K^2k^2}{\pi^2}\, \text{sn}^2 Kx/\pi = \dfrac{K}{\pi^2}(K-E) - 2\Sigma_1^\infty \dfrac{nq^n}{1-q^{2n}} \cos nx.$

Changing x into $x + \iota\pi K'/K$ we obtain

$$\frac{K^2}{\pi^2}\, \text{cosecam}^2\, Kx/\pi = \frac{K}{\pi^2}(K-E) - \Sigma_1^\infty \frac{n(1+q^{2n})}{1-q^{2n}} \cos nx.$$

Adding we obtain

$$\frac{K^2}{\pi^2}(k^2\, \text{sn}^2 Kx/\pi + \text{cosecam}^2\, Kx/\pi) = \frac{2K}{\pi^2}(K-E) - \Sigma_1^\infty \frac{n(1+q^n)}{1-q^n} \cos nx.$$

Also $\quad \Sigma n(1+q^n)\cos nx = -\tfrac{1}{4}\,\text{cosec}^2\, \tfrac{1}{2}x + \dfrac{q\{(1+q^2)\cos x - 2q\}}{(1-2q\cos x + q^2)^2}.$

Therefore

$$\Sigma_1^\infty \left\{\frac{n(1+q^n)}{1-q^n} - n(1+q^n)\right\} \cos nx = \frac{2K}{\pi^2}(K-E)$$

$$-\frac{K^2}{\pi^2}(k^2\, \text{sn}^2 Kx/\pi + \text{cosecam}^2\, Kx/\pi) + \tfrac{1}{4}\,\text{cosec}^2\, \tfrac{1}{2}x - \frac{q\{(1+q^2)\cos x - 2q\}}{(1-2q\cos x + q^2)^2} \ldots (3).$$

The required series is equal to the limit of the right-hand side of (3) when $x = 0$, that is

$$R = 4\pi c^2 \left\{\frac{2K}{\pi^2}(K-E) - \frac{q}{(1-q)^2}\right\}.$$

III. Professor Greenhill has kindly worked out the following investigation of the Motion under no forces of a Solid of Revolution in Infinite Liquid, by Weierstrass's functions.

Taking the expression (4) for the kinetic energy T of the solid of revolution and of the surrounding infinite frictionless liquid given in § 181, but writing p, q, r instead of ω_1, ω_2, ω_3 respectively, then

$$T = \tfrac{1}{2}P(u^2+v^2) + \tfrac{1}{2}Rw^2 + \tfrac{1}{2}A(p^2+q^2) + \tfrac{1}{2}Cr^2;$$

and employing this in the equations of motion of § 167, supposing there are no impressed forces; then since

$$\frac{dT}{du} = Pu, \quad \frac{dT}{dv} = Pv, \quad \frac{dT}{dw} = Rw;$$

$$\frac{dT}{dp} = Ap, \quad \frac{dT}{dq} = Aq, \quad \frac{dT}{dr} = Cr;$$

the equations of motion become

$$P\frac{du}{dt} - Pvr + Rwq = 0 \quad \ldots\ldots(1),$$

$$P\frac{dv}{dt} - Rwp + Pur = 0 \quad \ldots\ldots(2),$$

$$R\frac{dw}{dt} - Puq + Pvp = 0 \quad \ldots\ldots(3),$$

$$A\frac{dp}{dt} - (A-C)qr - (P-R)vw = 0 \quad \ldots\ldots(4),$$

$$A\frac{dq}{dt} + (A-C)pr + (P-R)uw = 0 \quad \ldots\ldots(5),$$

$$C\frac{dr}{dt} = 0 \quad \ldots\ldots(6).$$

Equation (6) shows that r is constant during the motion; and from the other equations we can obtain three first integrals of the equations of motion.

First, $P\left(u\frac{du}{dt} + v\frac{dv}{dt}\right) + Rw\frac{dw}{dt} + A\left(p\frac{dp}{dt} + q\frac{dq}{dt}\right) + Cr\frac{dr}{dt} = 0,$

so that $\tfrac{1}{2}P(u^2+v^2) + \tfrac{1}{2}Rw^2 + \tfrac{1}{2}A(p^2+q^2) + \tfrac{1}{2}Cr^2 = T \quad \ldots\ldots(7),$

a constant, the constant value of the kinetic energy during the motion.

Secondly, $P^2\left(u\frac{du}{dt} + v\frac{dv}{dt}\right) + R^2w\frac{dw}{dt} = 0,$

so that $P^2(u^2+v^2) + R^2w^2 = F^2 \quad \ldots\ldots(8),$

a constant; and then F represents the resultant linear momentum of the system.

Thirdly, $AP\left(\frac{du}{dt}p + u\frac{dp}{dt} + \frac{dv}{dt}q + v\frac{dq}{dt}\right) + CR\frac{dw}{dt}r = 0,$

so that $AP(up+vq) + CRwr = G \quad \ldots\ldots(9),$

a constant; and then G may be taken to represent the constant angular momentum of the system.

From equations (7), (8), (9),

$$P^2(u^2+v^2) = F^2 - R^2w^2,$$

$$A(p^2+q^2) = 2T - Cr^2 - Rw^2 - P(u^2+v^2),$$

$$= 2T - Cr^2 - \frac{F^2}{P} - \left(\frac{1}{R} - \frac{1}{P}\right)R^2w^2,$$

$$P(up+vq) = \frac{G - CRwr}{A};$$

so that from equation (3)

$$R^2\left(\frac{dw}{dt}\right)^2 = P^2(uq - vp)^2$$
$$= P^2\{(u^2+v^2)(p^2+q^2) - (up+vq)^2\}$$
$$= \left(\frac{F^2 - R^2w^2}{A}\right)\left\{2T - Cr^2 - \frac{F^2}{P} - \left(\frac{1}{R} - \frac{1}{P}\right)R^2w^2\right\} - \left(\frac{G - CRwr}{A}\right)^2,$$

a quartic function of Rw, so that Rw is an elliptic function of the time t, which we shall proceed to express by means of the notation of Weierstrass.

Putting, for the moment, $\frac{Rw}{F} = x = \cos\theta$, then

$$\left(\frac{dx}{dt}\right)^2 = \frac{F^2}{A}\left(\frac{1}{R} - \frac{1}{P}\right)(x - x_0)(x - x_1)(x - x_2)(x - x_3),$$

where x_0, x_1, x_2, x_3 denote the roots of the quartic in x, arranged in descending order of magnitude; also

$$x_0 + x_1 + x_2 + x_3 = 0.$$

Now put
$$x - x_0 = \frac{D}{s-d},$$

then
$$x - x_1 = \frac{D}{s-d}\frac{s-e_1}{d-e_1},$$
$$x - x_2 = \frac{D}{s-d}\frac{s-e_2}{d-e_2},$$
$$x - x_3 = \frac{D}{s-d}\frac{s-e_3}{d-e_3},$$

where e_1, e_2, e_3 are the roots of the discriminating cubic of the quartic

$$4s^3 - g_2 s - g_3 = 0,$$

g_2 and g_3 being the quadrinvariant and the cubinvariant.

Then
$$\left(\frac{ds}{dt}\right)^2 = \frac{F^2}{A}\left(\frac{1}{R} - \frac{1}{P}\right)D^2\frac{4s^3 - g_2 s - g_3}{4d^3 - g_2 d - g_3},$$

and we may choose D, so that

$$D^2 = 4d^3 - g_2 d - g_3,$$

and then
$$\left(\frac{ds}{dt}\right)^2 = \frac{F^2}{A}\left(\frac{1}{R} - \frac{1}{P}\right)(4s^3 - g_2 s - g_3),$$

so that now, with Weierstrass's notation (Halphen, *Traité des fonctions elliptiques et de leurs applications*, Paris, 1886),

$$s = \wp\,(t\omega_1/\tau + \omega_3),$$

ω_1 and ω_3 denoting the real and imaginary half periods of the elliptic functions, and τ the time of oscillation; the imaginary half period ω_3 being added in order to make s oscillate between e_2 and e_3, and therefore x between x_2 and x_3.

Then the time of oscillation τ is given by

$$\frac{\omega_1^2}{\tau^2} = \frac{F^2}{A}\left(\frac{1}{R} - \frac{1}{P}\right).$$

We may write $\mathrm{p}c$ instead of d; and use $\mathrm{p}u$ instead of $\mathrm{p}\,(t\omega_1/\tau + \omega_3)$ for brevity, and then $D = -\mathrm{p}'c$, and

$$x - x_0 = \frac{-\mathrm{p}'c}{\mathrm{p}u - \mathrm{p}c},$$

$$x - x_1 = \frac{-\mathrm{p}'c}{\mathrm{p}u - \mathrm{p}c}\,\frac{\mathrm{p}u - e_1}{\mathrm{p}c - e_1},$$

$$x - x_2 = \frac{-\mathrm{p}'c}{\mathrm{p}u - \mathrm{p}c}\,\frac{\mathrm{p}u - e_2}{\mathrm{p}c - e_2},$$

$$x - x_3 = \frac{-\mathrm{p}'c}{\mathrm{p}u - \mathrm{p}c}\,\frac{\mathrm{p}u - e_3}{\mathrm{p}c - e_3};$$

and then, as explained in the *Proceedings of the London Mathematical Society*, vol. XVII., p. 279, 1886, introducing the function ζu, defined by

$$\frac{d}{du}\zeta u = -\mathrm{p}u,$$

$$x_0 = 2\zeta c - \zeta 2c = -\tfrac{1}{2}\frac{\mathrm{p}''c}{\mathrm{p}'c},$$

$$x_1 = -\tfrac{1}{2}\frac{\mathrm{p}''(c + \omega_1)}{\mathrm{p}'(c + \omega_1)}, \quad x_2 = \ldots, \quad x_3 = \ldots,$$

and $\mathrm{p}2c$, $\mathrm{p}'2c$ are the coefficients of x^2 and x respectively in the quartic

$$(x - x_0)(x - x_1)(x - x_2)(x - x_3);$$

also

$$x = \zeta(u + c) - \zeta(u - c) - \zeta 2c = \tfrac{1}{2}\frac{\mathrm{p}'(u - c) - \mathrm{p}'2c}{\mathrm{p}(u - c) - \mathrm{p}2c}.$$

Taking the axis OZ in the direction of the resultant impulse F (fig. p. 166), then

$$Pu = -F\sin\theta\cos\phi, \quad Pv = F\sin\theta\sin\phi, \quad Rw = F\cos\theta,$$

and

$$P(up + vq) = F\sin\theta\,(-p\cos\phi + q\sin\phi),$$

$$= F\sin^2\theta\frac{d\psi}{dt};$$

so that equation (9) becomes

$$AF\sin^2\theta\,\frac{d\psi}{dt} = G - CRwr,$$

$$= G - CFr\cos\theta,$$

or, using x to denote $\cos\theta$,

$$\frac{d\psi}{dt} = \frac{G - CFrx}{AF(1 - x^2)} = \tfrac{1}{2}\,\frac{G + CFr}{AF}\,\frac{1}{1 + x} + \tfrac{1}{2}\,\frac{G - CFr}{AF}\,\frac{1}{1 - x},$$

the equation to determine the azimuthal motion ψ.

As explained in the *Proc. London Math. Soc.*, vol. XVII. p. 280, writing u for $t\omega_1/\tau + \omega_3$, this equation becomes

$$\frac{d\psi}{du} = \tfrac{1}{2}i \frac{\wp'a\,(\wp u - \wp c)}{(\wp a - \wp c)(\wp u - \wp a)} + \tfrac{1}{2}i \frac{\wp'b\,(\wp u - \wp c)}{(\wp b - \wp c)(\wp u - \wp b)},$$

a and b being the values of u which make $\cos\theta = -1$ or $+1$, respectively.

Then
$$2\frac{d\psi}{diu} = \frac{\wp'a}{\wp a - \wp c} + \frac{\wp'a}{\wp u - \wp a} - \frac{\wp'b}{\wp b - \wp c} - \frac{\wp'b}{\wp u - \wp b},$$
$$= \zeta(a+c) + \zeta(a-c) - 2\zeta a - \zeta(u+a) + \zeta(u-a) + 2\zeta a$$
$$+ \zeta(b+c) + \zeta(b-c) + 2\zeta b - \zeta(u+b) + \zeta(u-b) - 2\zeta b,$$

$$\psi = \tfrac{1}{2}i \log\frac{\sigma(u-a)\,\sigma(u-b)}{\sigma(u+a)\,\sigma(u+b)} + \tfrac{1}{2}iPu,$$

where
$$P = \zeta(a+c) + \zeta(a-c) + \zeta(b+c) + \zeta(b-c),$$

and
$$e^{i\psi} = e^{-\frac{1}{2}Pu}\sqrt{\frac{\sigma(u+a)\,\sigma(u+b)}{\sigma(u-a)\,\sigma(u-b)}}.$$

Taking a point on the axis OC at unit distance from O, the projection of the motion of this point on a plane through O perpendicular to OZ will be given by

$$x + iy = \sin\theta e^{i\psi},$$
$$= C\frac{\sigma(u+a)\,\sigma(u+b)}{\sigma(u+c)\,\sigma(u-c)}\exp(-\tfrac{1}{2}Pu).$$

In a similar manner, by means of the equation

$$\frac{d}{dt}\log(u+iv) = \tfrac{1}{2}\frac{d}{dt}\log(u^2+v^2) + i\frac{u\dot{v} - \dot{u}v}{u^2+v^2},$$
$$= \tfrac{1}{2}\frac{d}{dt}\log(u^2+v^2) - ir + i\frac{up+vq}{u^2+v^2}\frac{Rw}{P},$$
$$= \tfrac{1}{2}\frac{d}{dt}\log(F^2 - R^2w^2) - ir + i\frac{G - CRwr}{F^2 - R^2w^2}\frac{Rw}{A},$$

we can express $u + iv$ by means of Weierstrass's σ functions; and the same method can be applied to the expression of $p + iq$ and also of $x + iy$, x and y now denoting the coordinates of O with respect to fixed axes in a plane perpendicular to the direction of the resultant impulse F.

It will be noticed that the letter u has been used in two senses, first as expressing a component velocity of translation, and secondly as an abbreviation for $t\omega_1/\tau + \omega_3$; this was unavoidable in order to reconcile the different notations, but will not be found to lead to confusion.

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